Sveučilišni poslijediplomski doktorski studij matematike

Sveučilište u Zagrebu

Dragana Jankov

Integral expressions for series of functions of hypergeometric and Bessel types

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Supervisor: **Prof. dr. sc. Tibor Poganj**
Faculty od Maritime Studies, University of Rijeka

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Contents

Chapter 1

Introduction

In the 18th century, it became clear that the existing elementary functions are not sufficient to describe a number of unsolved problems in various branches of mathematics and physics. Functions that describe the new results were generally presented in the form of infinite series, integrals, or as solutions of differential equations. Some of them appeared more frequently, and because of the easier use, they were named, for example, Gamma, Beta function, Bessel functions, etc. That functions are collectively called *special functions*. One of the first issues about special functions is the set of four books, published between 1893 and 1902:

- J. Tannery, N. Molk, *Éléments de la théorie des fonctions elliptiques I, II: Calcul differentiel I, II. Applications*, Paris: Gauthier-Villars, 1893, 1896.
- J. Tannery, N. Molk, *Éléments de la théorie des fonctions elliptiques III, IV: Calcul integral I, II*, Paris: Gauthier-Villars, 1898, 1902.

that a total of $1148 + \mathrm{XXXII}$ pages long and contains previously known formulae with special functions.

A large part of this thesis deals with the integral representations and inequalities related to Bessel functions. G. N. Watson [130], in 1922, published the book, *A Treatise on the Theory of Bessel Functions*, which contains a wide range of results about Bessel functions, and had been useful to us, to obtain our own results.

There are three functional series with members containing Bessel functions of the first kind:

- *Neumann series*, i.e. the series in which order of the Bessel function of the first kind contains the current index of summation

$$\mathfrak{N}_\mu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\mu+n}(z), \qquad z \in \mathbb{C}\,. \tag{1.1}$$

Neumann series are named after the German mathematician Carl Gottfried Neumann [73], who in his book *Theorie der Besselschen Funktionen*, in 1867, studied only their special cases, namely those of integer order. A few years later, in 1887, Leopold Bernhard Gegenbauer [32, 33] expanded these series, having order the whole real line.

Neumann series are widely used. Especially interesting are the Neumann series of the zero–order Bessel function, i.e. series $\mathfrak{N}_0$, which appears as a relevant technical tool to solve the problem of infinite dielectric wedge through the Kontorovich–Lebedev transformation. It also occurs in the description of internal gravity waves in Bussinesq fluid, and in defining the properties of diffracted light beams. Wilkins [131] discussed the question of existence of an integral representation for a special Neumann series; Maximon [66] in 1956 represented a simple Neumann series $\mathfrak{N}_\mu$ appearing in the literature in connection with physical problems. Y. L. Luke [61] studied, in 1962, integral representation of Neumann series, for $\mu = 0$ and very recently Pogány and Süli [95] derived an integral representation of Neumann series $\mathfrak{N}_\mu(x)$, which approach helped us in performing a large number of results.

If we replace J_μ in (1.1) by modified Bessel function of the first kind I_μ, Bessel functions of the second kind Y_μ, K_μ (called Basset–Neumann and MacDonald functions, respectively), Hankel functions $H_\mu^{(1)}, H_\mu^{(2)}$ (or Bessel functions of the third kind) we get so–called *modified Neumann series*.

Neumann series, and also modified Neumann series will be discussed in Chapter 3.

- *Kapteyn series* are the series where the order of the Bessel function, and also the argument contains index of summation:

$$\mathfrak{K}_\mu(z) := \sum_{n=1}^{\infty} \alpha_n J_n\left((\mu+n)z\right), \qquad z \in \mathbb{C}.$$

Such series were introduced in 1893, by Willem Kapteyn [47], in his article *Recherches sur les functions de Fourier-Bessel.* These series have great applications in problems of mathematical physics. For example, a solution of famous Kepler's equation can be explicitly expressed by Kapteyn series of the first kind. Their application can be found in problems of pulsar physics, electromagnetic radiation, etc. There are also Kapteyn series of the second kind, which have been studied, in details, by Nielsen [79, 80], in 1901, and that series consist of the product of two Bessel functions of the first kind, of different orders. In 1906 Kapteyn [46] proved that every analytic function can be developed in Kapteyn series of the first kind. In Chapter 4, we will derive results related to this series.

- *Schlömilch series* appear when the argument contains the current index of summation, i.e. the series of the form:

$$\mathfrak{S}_\mu(z) := \sum_{n=1}^{\infty} \alpha_n J_\mu\left((\mu+n)z\right), \quad z \in \mathbb{C}.$$

Oscar Xaver Schlömilch [109] was the first who defined that series, in 1857, in the article *Über die Bessel'schen Function*, but he looked only at cases when the series contains of Bessel functions of the first kind, of order $\mu = 0, 1$. Their use is so widespread in the field of physics, such as the use of Kapteyn series. Rayleigh [99], in 1911, pointed out that in the case $\mu = 0$ these series are useful in the study of periodic transverse vibrations of two–dimensional membranes. Generalized Schlömilch series appeared in the Nielsen's memoirs [74, 75, 76, 77, 78, 79, 80] from 1899, 1900 and 1901. Filon [29] in 1906 first studied the possibility of development of arbitrary function in generalized Schlömilch series. Integral representations for that series are established in Chapter 5.

In this thesis we also deal with Hurwitz–Lerch Zeta function, which is discussed in Chapter 6.

A general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ is first defined by Erdélyi, Magnus, Oberhettinger and Tricomi, in 1953 (see, e.g. [28, p. 27, Eq. 1.11 (1)]):

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

where $a \in \mathbb{Z}\backslash\mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$ and contains a whole number of special functions, as special cases, such as Riemann Zeta function $\zeta(s)$, Hurwitz (or generalized) Zeta function $\zeta(s, a)$, Lerch Zeta function $\ell_s(\xi)$, Polylogarithmic and Lipschitz–Lerch Zeta functions and their generalizations that were observed by Goyal and Laddha [34], Garg, Jain and Kalla [30, 31], Lin and Srivastava [60], etc., starting from 1997, till now.

Extended general Hurwitz–Lerch Zeta function was introduced in article

- M. Garg, K. Jain, S. L. Kalla, A further study of general Hurwitz–Lerch Zeta function, *Algebras Groups Geom.* **25(3)** (2008), 311–319.

in the following form:

$$\Phi_{\alpha,\beta;\gamma}(z, s, r) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \frac{z^n}{(n+r)^s},$$

where $\gamma, r \notin \mathbb{Z}_0^-$, $s \in \mathbb{C}, \Re(s) > 0$ when $|z| < 1$ and $\Re(\gamma + s - \alpha - \beta) > 0$ when $|z| = 1$.

Our main aim is to determine the integral representations, and two–sided inequalities for the extended general Hurwitz–Lerch Zeta function, as well as for the extended Hurwitz–Lerch Zeta function.

E. L. Mathieu [65], 1890, studied the so–called Mathieu series. These series and their many generalizations, have become the subject of many investigations in recent years. In our work we use Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series, which was introduced in 2005, by Pogány [91], in response to an open question, posed by Feng Qi [96], four years before.

In integral representation of extended general Hurwitz–Lerch Zeta function, we will use Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series, first time mentioned in the article

- T. K. Pogány, Integral representation of Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series, *Integral Transforms Spec. Funct.* **16(8)** (2005), 685–689.

Inequalities and integral representations of Mathieu series were also studied by Cerone and Lenard [15], in 2003, Qi [96], in 2001, Srivastava and Tomovski [121], in 2004 and Pogány *et al.* [94], in 2006, while multiple Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series were considered by Draščić Ban [23, 24], in her doctoral dissertation, in 2009, and in one article, in 2010. In this thesis are used some Pogány's integral representation for $(\mathbf{a}, \boldsymbol{\lambda})$–series, for deriving some new inequalities for the extended Hurwitz–Lerch Zeta function.

Then, by using the above described integral representation, we shall derive two newly, two-sided inequalities for the extended general Hurwitz–Lerch Zeta function.

As a generalization, in the dissertation is also examined the extended Hurwitz–Lerch Zeta function, $\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z, s, a)$, which can be found in the article

- H. M. Srivastava, R. K. Saxena, T. K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch Zeta function, *Integral Transforms Spec. Funct.* **22** (2011), 487–506.

and it is obtained by introducing Fox-Wright generalized hypergeometric functions in the kernel of Hurwitz–Lerch Zeta function:

$$\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a) = \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\,\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\,\frac{z^n}{(n+a)^s}\,.$$

Here $p,q\in\mathbb{N}_0$; $\lambda_j\in\mathbb{C}$, $j=1,\cdots,p$; $a,\mu_j\in\mathbb{C}\setminus Z_0^-$, $j=1,\cdots,q$; $\rho_j,\sigma_k\in\mathbb{R}_+$, $j=1,\cdots,p$; $k=1,\cdots,q$; $\Delta>-1$ when $s,z\in\mathbb{C}$; $\Delta=-1$ and $s\in\mathbb{C}$ when $|z|<\nabla$; $\Delta=-1$ and $\Re(\Xi)>\frac{1}{2}$ when $|z|=\nabla$, where

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j > -1\,,\quad \nabla := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right)\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right) \text{ and } \Xi := s+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}\,.$$

It is worth to mention that until now there were not known inequalities for the Hurwitz–Lerch Zeta function.

At the end of the Chapter 6, we will define certain new incomplete generalized Hurwitz–Lerch Zeta functions and incomplete generalized Gamma functions and we shall also investigate their important properties.

Remark 1.1. It is important to mention that some useful labels, which can be found in this chapter, are described at the beginning of the next chapter, in Section 2.1. ■

Some results of these dissertation have been published, or accepted for publication in the form of the following articles:

- Á. Baricz, D. Jankov, T.K. Pogány, Integral representations for Neumann-type series of Bessel functions I_ν, Y_ν and K_ν, *Proc. Amer. Math. Soc.* (2011) (to appear).
- Á. Baricz, D. Jankov, T. K. Pogány, Integral representation of first kind Kapteyn series, *J. Math. Phys.* **52(4)** (2011), Art. 043518, pp. 7.
- Á. Baricz, D. Jankov, T. K. Pogány, On Neumann series of Bessel functions, *Integral Transforms Spec. Funct.* (2011) (to appear).
- D. Jankov, T. K. Pogány, R. K. Saxena, An extended general Hurwitz–Lerch Zeta function as a Mathieu (a,λ) – series, *Appl. Math. Lett.* **24(8)** (2011), 1473–1476.
- D. Jankov, T. K. Pogány, E. Süli, On the coefficients of Neumann series of Bessel functions, *J. Math. Anal. Appl.* **380(2)** (2011), 628–631.
- R. K. Saxena, T. K. Pogány, R. Saxena, D. Jankov, On generalized Hurwitz-Lerch Zeta distributions occuring in statistical inference, *Acta Univ. Sapientiae Math.* **3(1)** (2011), 43–59.

- H. M. Srivastava, D. Jankov, T. K. Pogány, R. K. Saxena, Two-sided inequalities for the extended Hurwitz–Lerch Zeta function, *Comput. Math. Appl.* **62(1)** (2011), 516–522.

The most important parts of the dissertation were exposed at the Seminar on Optimization and Applications in Osijek, and on the Seminar on Inequalities and Applications, in Zagreb, in 2011 (in three occasions).

A part of the dissertation, about generalized Hurwitz-Lerch Zeta distribution, was held at the Second Conference of the Central European Network, in Zürich, 2011.

Chapter 2

Basic definitions

Before stating the main results of this thesis, we outline the basic concepts, definitions and results which would be necessary in proving our own findings. Let us first introduce some useful labels.

2.1 Some useful labels

Below, we would need some usual labels like

- $\mathbb{N} = \{1, 2, 3, \dots\}$, which is the set of natural numbers and
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ stands for the set of all integers and
- $\mathbb{Z}_0^- = \{\dots, -2, -1, 0\}$, $\mathbb{Z}_0^+ = \{0, 1, 2, \dots\}$.

Further, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand for the sets of real, *positive* real and complex numbers, respectively.

Throughout in this thesis, by convention, $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional part of some real number a, respectively.

We would also use a symbol for the imaginary unit i, which is defined solely by the property that its square is -1, i.e. $i = \sqrt{-1}$.

It is important to mention, that we will use symbol $\square$ for the end of the proof, and $\blacksquare$ for the end of the Remark.

2.2 The Gamma function

The Gamma function has caught the interest of some of the most prominent mathematicians of all times. Its history, notably documented by Philip J. Davis in an article that won him the Chauvenet Prize, in 1963, reflects many of the major developments within mathematics since the 18th century. In his article

- P. J. Davis, Leonhard Euler's integral: A historical profile of the gamma function, *Amer. Math. Monthly* **66(10)** (1959), 849-869.

Davis wrote: *"Each generation has found something of interest to say about the Gamma function. Perhaps the next generation will also"*.

In this section, we study major properties of the Gamma function and introduce some other functions which can be expressed in terms of the Gamma function, namely Psi and Beta function and also the Pochhammer symbol.

Gamma function is defined by a definite integral due to Leonhard Euler

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt\,, \qquad \Re(z) > 0\,. \tag{2.1}$$

The notation $\Gamma(z)$ is due to French mathematician Adrien–Marie Legendre.

Using integration by parts, from (2.1) we can easily get

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}\,, \qquad z > 0\,. \tag{2.2}$$

That relation is called the *recurrence formula* or *recurrence relation* of the Gamma function. For $z = n \in \mathbb{N}$, from (2.2) it follows that

$$\Gamma(n) = (n-1)!\,.$$

The recurrence relation is not the only functional equation satisfied by the Gamma function. Another important property is the *reflection formula*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

which gives relation between the Gamma function of positive and negative numbers.

For $z = \frac{1}{2}$, from the previous equation, it follows that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,.$$

In examining the convergence conditions of corresponding series of Bessel functions of the first kind, we would need the formula for asymptotic behavior of the Gamma function, for large values of z:

$$\Gamma(z) = \sqrt{2\pi} e^{-z} z^{z-1/2} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)\,, \qquad z > 0\,, \tag{2.3}$$

such that usually write

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-1/2}, \qquad |z| \to \infty\,.$$

In (2.3) we have familiar so–called *order symbol* $\mathcal{O}$.

Gamma function also has the following properties (see [98, p. 9]):

- $\Gamma(z)$ is analytic except at nonpositive integers, and when $z = \infty$;
- $\Gamma(z)$ has a simple pole at each nonpositive integer, $z \in \mathbb{Z}_0^-$;
- $\Gamma(z)$ has an essential singularity at $z = \infty$, a point of condensation of poles;
- $\Gamma(z)$ is never zero, because $1/\Gamma(z)$ has no poles.

2.2.1 Psi (or Digamma) function

Psi (or Digamma) function $\psi(z)$ is defined as the logarithmic derivative of the gamma function:

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t)\, dt\,.$$

We can express $\psi(z)$ [1, Eq. (6.3.16)] as follows (see also [115, p. 14, Eq. 1.2(3)]:

$$\psi(z) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k-1} \right) - \mathbf{C}\,, \qquad z \in \mathbb{C} \setminus \mathbb{Z}_0^-\,,$$

where $\mathbf{C}$ denotes the celebrated Euler-Mascheroni constant given by

$$\mathbf{C} := \lim_{n\to\infty} (H_n - \log n) \approx 0.5772\,,$$

where H_n are called the harmonic numbers defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k}\,, \qquad n \in \mathbb{N}.$$

Finally, let us remark that the Digamma function $\psi(x)$ increases on its entire range and possesses the unique positive nil $\alpha_0 = \psi^{-1}(0) \approx 1.4616$. One of the useful properties of the Digamma function is that

$$\psi(z+1) = \frac{1}{z} + \psi(z), \qquad z > 0\,.$$

2.2.2 The Beta function

The Beta function, also called the *Euler integral of the first kind*, is a special function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad \Re(x), \Re(y) > 0\,.$$

The Beta function is intimately related to the Gamma function, which is described in [98, p. 18]:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \qquad \Re(x), \Re(y) > 0\,. \tag{2.4}$$

Using (2.4) it easily follows that Beta function is invariant with respect to parameter permutation, meaning that

$$B(x, y) = B(y, x)\,.$$

2.2.3 The Pochhammer symbol

The Pochhammer symbol (or the *shifted factorial*), introduced by Leo August Pochhammer, is defined, in terms of Euler's Gamma function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mu = 0;\ \lambda \in \mathbb{C}\setminus\{0\} \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } \mu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases},$$

it being understood *conventionally* that $(0)_0 := 1$.

The Pochhammer symbol also satisfies

$$(-\lambda)_\mu = (-1)^\mu\,(\lambda-\mu+1)_\mu\,, \qquad \mu \in \mathbb{N}_0\,. \tag{2.5}$$

2.3 Dirichlet series

One of our main mathematical tools is the series

$$\mathcal{D}_{\mathbf{a}}(s) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}\,, \qquad s > 0\,, \tag{2.6}$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \qquad \text{as} \qquad n \to \infty\,.$$

That are *Dirichlet series on the* λ_n*–type*. For $\lambda_n = n$, (2.6) becomes power series

$$\mathcal{D}_{\mathbf{a}}(s) := \sum_{n=1}^{\infty} a_n e^{-ns}\,, \qquad s > 0$$

and for $\lambda_n = \ln n$, we have series of the form

$$\mathcal{D}_{\mathbf{a}}(s) := \sum_{n=1}^{\infty} a_n n^{-s}\,, \qquad s > 0\,,$$

which is called *ordinary Dirichlet series*.

In this thesis we mostly deal with series of the form (2.6), where s is real variable. We also need a variant of closed integral form representation of Dirichlet series, which is derived below, following mainly [39], [48**C. §V**]. The heart of the matter is the known Stieltjes integral formula

$$\int_a^b f(x)\,dA_s(x) = \sum_{n=1}^{\infty} f(\lambda_n)(A_s(\lambda_n+) - A_s(\lambda_n-)), \tag{2.7}$$

such that is valid for A_s-integrable f, where the *step function*

$$A_s(x) := \sum_{n:\ \lambda_n \le x} (A_s(\lambda_n+) - A_s(\lambda_n-)) \tag{2.8}$$

possesses the discontinuity set $\{\lambda_n\}$. Assuming that $\lambda(x)$ is monotone increasing positive function such that runs to the infinity with growing x and it is $\{\lambda_n\}_{n\in\mathbb{N}} = \lambda(x)\big|_{\mathbb{N}}$, we deduce that λ is invertible with the unique inverse λ^{-1}. Now, putting $a_n =: A_s(\lambda_n+) - A_s(\lambda_n-)$ into (2.8) we get

$$A_s(x) = \sum_{n:\ \lambda_n \le x} a_n = \sum_{n=1}^{[\lambda^{-1}(x)]} a_n .$$

Here $A_s(x)$ is the function such that has jump of magnitude a_n at λ_n, $n \in \mathbb{N}$. So, taking $f(x) = e^{-sx}$ having in mind that $[a,b] = [0,x]$, by (2.7) we deduce

$$\sum_{n:\ \lambda_n \le x} a_n e^{-\lambda_n s} = \int_0^x e^{-s\,t} dA_s(t). \tag{2.9}$$

Letting $x \to \infty$ in (2.9) we obtain an integral such that is equiconvergent with $\mathcal{D}_{\mathbf{a}}(s)$, so

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \int_0^{\infty} e^{-s\,t} dA_s(t)\,, \qquad s > 0\,. \tag{2.10}$$

Now, the integration by part results in a Laplace integral instead of the Laplace–Stieltjes integral (2.10). Indeed, as e^{-sx} decreases in x being s positive, taking $a(0) = 0$, the convergence of the Laplace–Stieltjes integral (2.10) ensures the validity of the desired relation

$$\mathcal{D}_{\mathbf{a}}(s) = s \int_0^{\infty} e^{-s\,t} A_s(t)\, dt\,, \qquad s > 0\,. \tag{2.11}$$

2.3.1 Euler–Maclaurin summation formula

Euler-Maclaurin formula provides a powerful connection between integrals and sums. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series using integrals and the machinery of calculus.

The formula was discovered independently by Leonhard Euler and Colin Maclaurin around 1735. Euler needed it to compute slowly converging infinite series, while Maclaurin used it to calculate integrals. Their famous summation formula, of the first degree is

$$\sum_{n=k}^{l} a_n = \int_k^l a(x)dx + \frac{1}{2}(a_l + a_k) + \int_k^l a'(x)B_1(x)\, dx,$$

where $B_1(x) = \{x\} - \frac{1}{2}$ is the first degree Bernoulli polynomial. It generally holds [34, p. 72, Eq. (1)]

$$\sum_{n=k}^{\ell} f(n) = \int_k^{\ell} f(x)dx + \tfrac{1}{2}\big(f(k) + f(\ell)\big) + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\Big(f^{(2j-1)}(\ell) - f^{(2j-1)}(k)\Big)$$
$$- \int_k^{\ell} \frac{B_{2m}(x)}{(2m)!}\, f^{(2m)}(x)\, dx\,, \qquad m \in \mathbb{N},$$

where $B_p(x) = (x + B)^p$, $0 \le x < 1$ represents Bernoulli polinomial of order $p \in \mathbb{N}$, while B_k are appropriate Bernoulli numbers. On $[\ell, \ell + 1), \ell \in \mathbb{N}$, $B_p(x)$ are periodic with period 1.

Summation formulae, of the first kind ($p = 1$) we will use in condensed form, under the condition $a \in C^1[k, l]$, $k, l \in \mathbb{Z}$, $k < l$:

$$\sum_{n=k+1}^{l} a_n = \int_k^{l} (a(x) + \{x\}a'(x))dx \equiv \int_k^{l} \mathfrak{d}_x a(x)\, dx\,, \tag{2.12}$$

where

$$\mathfrak{d}_x := 1 + \{x\}\frac{\partial}{\partial x}\,,$$

see [94, 95].

Finally, assuming that $\mathbf{a} := a(x)\big|_{\mathbb{N}}$, $a \in C^1[0, \infty)$ we summ up $A_s(t)$ by the Euler-Maclaurin formula completing the desired closed form integral representation of Dirichlet series $\mathcal{D}_{\mathbf{a}}(s)$ without any sums.

The articles [89, 90, 91] contain certain special cases of (2.11) specifying $a_n = 1$; λ, a are powers of the same monotonous increasing sequence etc.

The multiple Euler–Maclaurin summation formulae are discussed in detail e.g. in [23].

2.4 Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series

The so–called Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series

$$\mathfrak{M}_s(\mathbf{a}, \boldsymbol{\lambda}; r) = \sum_{n=0}^{\infty} \frac{a_n}{(\lambda_n + r)^s}\,, \qquad r, s > 0\,, \tag{2.13}$$

has been introduced by Pogány [89], giving an exhaustive answer to an Open Problem posed by Qi [96], deriving closed form integral representation and bilateral bounding inequalities for $\mathfrak{M}_s(\mathbf{a}, \boldsymbol{\lambda}; r)$, generalizing at the same time some earlier results by Cerone and Lenard [15], Qi [96], Srivastava and Tomovski [121] and others.

The mentioned Pogány's integral representation formula for Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series is [89, Theorem 1]:

$$\mathfrak{M}_s(\mathbf{a}, \boldsymbol{\lambda}; r) = \frac{a_0}{r^s} + s\int_{\lambda_1}^{\infty}\int_0^{[\lambda^{-1}(x)]} \frac{a(u) + a'(u)\{u\}}{(r + x)^{s+1}}\, dx\, du\,, \tag{2.14}$$

where $a \in C^1[0, \infty)$ and $a\big|_{\mathbb{N}_0} \equiv \mathbf{a}$, λ^{-1} stands for the inverse of λ and the series (2.13) converges.

The series (2.13) is assumed to be convergent and the sequences $\mathbf{a} := (a_n)_{n\in\mathbb{N}_0}, \boldsymbol{\lambda} := (\lambda_n)_{n\in\mathbb{N}_0}$ are positive. Following the convention that (λ_n) is monotone increasing divergent, we have

$$\boldsymbol{\lambda}: \quad 0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n \xrightarrow[n\to\infty]{} \infty\,.$$

2.5 Hypergeometric and generalized hypergeometric functions

Hypergeometric functions form an important class of special functions. They were introduced in 1866, by C. F. Gauss and after that have proved to be of enormous significance in mathematics and the mathematical sciences elsewhere. Here, we describe some properties of hypergeometric functions which are useful for us to derive some of our main results.

2.5.1 Gaussian hypergeometric function

Gaussian hypergeometric function is the power series

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\frac{z^k}{k!} = 1 + \frac{ab}{c}\,z + \frac{a(a+1)\,b(b+1)}{c(c+1)}\,\frac{z^2}{2} + \ldots, \tag{2.15}$$

where z is a complex variable, a, b and c are real or complex parameters and $(a)_k$ is the Pochhammer symbol.

The series is not defined for $c = -m$, $m \in \mathbb{N}_0$, provided that a or b is not the negative integer n such that $n < m$. Furthermore, if the series (2.15) is defined but a or b is equal to $(-n)$, $n \in \mathbb{N}_0$, then it terminates in a finite number of terms and its sum is then the polynomial of degree n in variable z. Except for this case, in which the series is absolutely convergent for $|z| < \infty$, the domain of absolute convergence of the series (2.15) is the unit disc, i.e. $|z| < 1$. In this case it is said that the series (2.15) defines the Gaussian or hypergeometric function

$$g(z) = {}_2F_1(a,b;c;z). \tag{2.16}$$

Also, on the unit circle $|z| = 1$, the series in (2.15) converges absolutely when $\Re(c-a-b) > 0$, converges conditionally when $-1 < \Re(c-a-b) \le 0$ apart from at $z = 1$, and does not converge if $\Re(c-a-b) \le -1$.

It can be verified [113, p. 6] that the function $g(z)$ is the solution of the second order differential equation

$$z(1-z)g''(z) + (c - (a+b+1)z)\,g'(z) - ab\,g(z) = 0, \tag{2.17}$$

in the region $|z| < 1$. However, the function (2.16) can be analytically continued to the other parts of the complex plane, i.e. solutions of the equation (2.17) are also defined outside the unit circle. These solutions are provided by following substitutions in the equation (2.17):

- substitution $z = 1 - y$ yields solutions valid in the region $|1 - z| < 1$,
- substitution $z = 1/y$ yields solutions valid in the region $|z| > 1$.

2.5.2 Generalized hypergeometric function

For b_i $(i = 1, 2, \dots, q)$ non negative integers or zero, the series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}$$

is called a *generalized hypergeometric series* (see [69]) and is denoted by the symbols

$${}_pF_q[z] = {}_pF_q[(a_p); (b_q)|z] = {}_pF_q\left[\begin{array}{c} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{array} \middle| z \right].$$

When $p \leq q$, the generalized hypergeometric function converges for all complex values of z; thus, ${}_pF_q[z]$ is an entire function. When $p > q + 1$, the series converges only for $z = 0$, unless it terminates (as when one of the parameters a_i is a negative integer) in which case it is just a polynomial in z. When $p = q+1$, the series converges in the unit disk $|z| < 1$, and also for $|z| = 1$ provided that

$$\Re\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i \right) > 0 \,.$$

The complex members of the sequences (a_p), (b_q) are called *parameters* and z is the *argument* of the function.

2.5.3 Fox–Wright generalized hypergeometric function

In this thesis, we also need the Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*[\cdot]$ with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, which is defined by [49, p. 56]

$${}_p\Psi_q^*\left[\begin{array}{c} (a_1, \rho_1), \cdots, (a_p, \rho_p) \\ (b_1, \sigma_1), \cdots, (b_q, \sigma_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (a_j)_{\rho_j n}}{\prod\limits_{j=1}^{q} (b_j)_{\sigma_j n}} \frac{z^n}{n!}, \tag{2.18}$$

where a_j, $b_k \in \mathbb{C}$ and ρ_j, $\sigma_k \in \mathbb{R}_+$, $j = 1, \cdots, p$; $k = 1, \cdots, q$. The defining series in (2.18) converges in the whole complex z-plane when

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j > -1\,; \tag{2.19}$$

when $\Delta = 0$, then the series in (2.18) converges for $|z| < \nabla$, where

$$\nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right). \tag{2.20}$$

If, in the definition (2.18), we set $\rho_1 = \cdots = \rho_p = 1$ and $\sigma_1 = \cdots = \sigma_q = 1$, we get generalized hypergeometric function ${}_pF_q[\cdot]$.

2.6 Bessel differential equation

The Bessel differential equation is the linear second–order ordinary differential equation given by

$$x^2 \frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} + (x^2 - \nu^2)y = 0\,, \qquad \nu \in \mathbb{C}\,. \tag{2.21}$$

The solutions to this equation define the Bessel function of the first kind J_ν and Bessel function of the second kind Y_ν. The equation has a regular singularity at zero, and an irregular singularity at infinity.

The function $J_\nu(x)$ is defined by the equation

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}\,.$$

For $\nu \notin \mathbb{Z}$, functions $J_\nu(x)$ i $J_{-\nu}(x)$ are linearly independent, and so the solutions of differential equation (2.21) are independent, while for $\nu \in \mathbb{Z}$ it holds

$$J_{-\nu}(x) = (-1)^\nu J_\nu(x).$$

Bessel functions of first kind, which were introduced by the Swiss mathematician Daniel Bernoulli, 1740, in his paper *Demonstrationes theorematum suorum de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae*, represent the general solution of the homogeneous Bessel differential equation of the second degree. Alexandre S. Chessin [13, 14] was a first who gave an explicit solutions of Bessel differential equation with general nonhomogeneous part, in 1902.

We are also interested in estimates for Bessel function of the first kind. Landau [53] gave the following bounds for Bessel function $J_\nu(x)$:

$$|J_\nu(x)| \le b_L \nu^{-1/3}, \qquad b_L := \sqrt[3]{2}\sup_{x\in\mathbb{R}_+}\big(\mathrm{Ai}(x)\big) \tag{2.22}$$

and

$$|J_\nu(x)| \le c_L |x|^{-1/3}, \qquad c_L := \sup_{x\in\mathbb{R}_+} x^{1/3}\big(J_0(x)\big), \tag{2.23}$$

where $\mathrm{Ai}(x)$ stands for the familiar Airy function, which is solution of differential equation

$$y'' - xy = 0, \qquad y = \mathrm{Ai}(x)$$

and can be expressed as

$$\mathrm{Ai}(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\left(J_{-(1/3)}\left(2\,(x/3)^{3/2}\right) + J_{1/3}\left(2\,(x/3)^{3/2}\right)\right)\,.$$

Olenko [85] also gave sharp upper bound for Bessel function:

$$\sup_{x\ge 0}\sqrt{x}|J_\nu(x)| \le b_L\sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} =: d_0\,, \qquad \nu > 0,$$

where α_1 is the smallest positive zero of Airy's function $\mathrm{Ai}(x)$, and b_L is the first Landau's constant.

There is also Krasikov's [50] uniform bound:

$$J_\nu^2(x) \le \frac{4(4x^2-(2\nu+1)(2\nu+5))}{\pi((4x^2-\mu)^{3/2}-\mu)}\,, \qquad x > \sqrt{\mu+\mu^{2/3}}\,,\;\; \nu > -1/2 \tag{2.24}$$

where $\mu = (2\nu+1)(2\nu+3)$. This bound is sharp in the sense that

$$J_\nu^2(x) \ge \frac{4(4x^2-(2\nu+1)(2\nu+5))}{\pi((4x^2-\mu)^{3/2}-\mu)}$$

in all points between two consecutive zeros of Bessel function $J_\nu(x)$ [50, Theorem 2]. Krasikov also pointed out that estimations (2.22) and (2.23) are sharp only for values that are in the neighborhood of the smallest positive zero $j_{\nu,1}$ of the Bessel function $J_\nu(x)$, while his estimate (2.24) gives sharp upper bound in whole area.

Pogány and Srivastava [118, p. 199, Eq. (19)] proposed a better, hybrid estimator:

$$|J_\nu(x)| \le \mathfrak{W}_\nu(x) := \frac{d_O}{\sqrt{x}}\,\chi_{(0,A_\lambda]}(x) + \sqrt{\mathfrak{K}_\nu(x)}\,\big(1-\chi_{(0,A_\lambda]}(x)\big)\,,$$

where

$$\mathfrak{K}_\nu(x) := \frac{4(4x^2-(2\nu+1)(2\nu+5))}{\pi((4x^2-\mu)^{3/2}-\mu)},$$

while

$$A_\lambda = \frac{1}{2}\left(\lambda + (\lambda+1)^{2/3}\right).$$

In this thesis, we shall use Landau's bounds, because of their simplicity. Derived results one can expand using hybrid estimator $\mathfrak{W}_\nu$ as well.

Further, exponential bounding inequalities for $J_\nu(x)$ are published by Pogány [92] and Sitnik [112].

2.7 The Struve function

In determining the integral representation of the second kind Neumann series $\mathfrak{X}_\nu(z)$, which will be introduced in Chapter 3, (3.27), we need the less mentioned *Struve function.* Struve function $\mathbf{H}_\nu(z)$, of order ν, is defined by the relations [130, p. 328]

$$\begin{aligned}\mathbf{H}_\nu(z) &= \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\sin(zt)\,dt\\ &= \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\sin(z\cos\theta)\sin^{2\nu}\theta\,d\theta\,,\end{aligned}$$

provided that $\Re(\nu) > -\frac{1}{2}$.

It also holds

$$\mathbf{H}_\nu(z) = \sum_{n=0}^\infty \frac{(-1)^n\left(\frac{z}{2}\right)^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}\,.$$

The function $\mathbf{H}_\nu(z)$ is defined by this equation for all values of ν, whether $\Re(\nu)$ exceeds $-\frac{1}{2}$ or not.

2.8 Fractional differintegral

In order to solve the nonhomogeneous Bessel differential equation, we will use *fractional derivation* and *fractional integration*, i.e. *fractional differintegration*.

So, let us first introduce, according to [58, p. 1488, Definition], the fractional derivative and the fractional integral of order ν of some suitable function f, see also [59, 127, 128, 129].

If the function $f(z)$ is analytic (regular) inside and on $\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}$, where $\mathcal{C}$ is a contour along the cut joining the points z and $-\infty + i\Im\{z\}$, which starts from the point at $-\infty$, encicles the point z once counter–clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\Im\{z\}$, which starts from the point at ∞, encircles the point z once counter–clockwise, and returns to the point at ∞,

$$f_\mu(z) = (f(z))_\mu := \frac{\Gamma(\mu+1)}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{\mu+1}}\, d\zeta$$

for all $\mu \in \mathbb{R} \setminus \mathbb{Z}^-$; $\mathbb{Z}^- := \{-1, -2, -3, \cdots\}$ and

$$f_{-n}(z) := \lim_{\mu \to -n} \left(f_\mu(z)\right)_\mu , \qquad n \in \mathbb{N},$$

where $\zeta \neq z$,

$$-\pi \leq \arg(\zeta - z) \leq \pi\,, \qquad \text{for } \mathcal{C}^-,$$

and

$$0 \leq \arg(\zeta - z) \leq 2\pi\,, \qquad \text{for } \mathcal{C}^+,$$

then $f_\mu(z)$, $\mu > 0$ is said to be the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$, $\mu < 0$ is said to be the fractional integral of $f(z)$ of order $-\mu$, provided that

$$|f_\mu(z)| < \infty, \qquad \mu \in \mathbb{R}.$$

At this point let us recall that the fractional differintegral operator (see e.g. [82, 83, 58])

- is linear, i.e. if the functions $f(z)$ and $g(z)$ are single–valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then

  $$(k_1 f(z) + k_2 g(z))_\nu = k_1 f_\nu(z) + k_2 g_\nu(z)\,, \qquad \nu \in \mathbb{R}, z \in \Omega$$

  for any constants k_1 and k_2;

- preserves the index law: If the function $f(z)$ is single–valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then

  $$(f_\mu(z))_\nu = f_{\mu+\nu}(z) = (f_\nu(z))_\mu\,,$$

  where $f_\mu(z) \neq 0$, $f_\nu(z) \neq 0$, $\mu, \nu \in \mathbb{R}$, $z \in \Omega$;

- permits the generalized Leibniz rule: If the functions $f(z)$ and $g(z)$ are single–valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then

$$(f(z) \cdot g(z))_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} f_{\nu-n}(z) \cdot g_n(z) , \qquad \nu \in \mathbb{R},\, z \in \Omega,$$

where $g_n(z)$ is the ordinary derivative of $g(z)$ of order $n \in \mathbb{N}_0$, it being tacitly assumed that $g(z)$ is the polynomial part (if any) of the product $f(z) \cdot g(z)$.

Fractional differintegral operator also has the following properties:

- For a constant λ,

$$\left(e^{\lambda z}\right)_\nu = \lambda^\nu e^{\lambda z} , \qquad \lambda \neq 0,\, \nu \in \mathbb{R},\, z \in \mathbb{C};$$

- For a constant λ,

$$\left(e^{-\lambda z}\right)_\nu = e^{-i\pi\nu} \lambda^\nu e^{\lambda z} , \qquad \lambda \neq 0,\, \nu \in \mathbb{R},\, z \in \mathbb{C};$$

- For a constant λ,

$$\left(z^\lambda\right)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu - \lambda)}{\Gamma(-\lambda)} z^{\lambda-\nu} , \qquad \nu \in \mathbb{R},\, z \in \mathbb{C}, \left| \frac{\Gamma(\nu - \lambda)}{\Gamma(-\lambda)} \right| < \infty .$$

Chapter 3

Neumann series

The series

$$\mathfrak{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+n}(z), \qquad z \in \mathbb{C}, \tag{3.1}$$

where ν, α_n are constants and J_μ signifies the Bessel function of the first kind of order μ, is called a *Neumann series* [130, Chapter XVI]. Such series owe their name to the fact that they were first systematically considered (for integer μ) by Carl Gottfried Neumann in his important book [73] in 1867, and subsequently in 1877, Leopold Bernhard Gegenbauer extended such series to $\mu \in \mathbb{R}$ (see [130, p. 522]).

Neumann series of Bessel functions arise in a number of application areas. For example, in connection with random noise, Rice [100, Eqs. (3.10–17)] applied Bennett's result

$$\sum_{n=1}^{\infty} \left(\frac{v}{a}\right)^n J_n(ai\,v) = e^{v^2/2} \int_0^v x e^{-x^2/2} J_0(ai\,x)\,dx\,. \tag{3.2}$$

Luke [61, p. 271–288] proved that

$$1 - \int_0^v e^{-(u+x)} J_0\left(2i\sqrt{ux}\right) dx = \begin{cases} e^{-(u+v)} \displaystyle\sum_{n=0}^{\infty} \left(\frac{u}{v}\right)^{n/2} J_n\left(2i\sqrt{uv}\right), & \text{if } u < v \\ 1 - e^{-(u+v)} \displaystyle\sum_{n=0}^{\infty} \left(\frac{v}{u}\right)^{n/2} J_n\left(2i\sqrt{uv}\right), & \text{if } u > v \end{cases}.$$

In both of these applications $\mathfrak{N}_0$ plays a key role. The function $\mathfrak{N}_0$ also appears as a relevant technical tool in the solution of the infinite dielectric wedge problem by Kontorovich–Lebedev transforms [101]. It also arises in the description of internal gravity waves in a Boussinesq fluid [72], as well as in the study of the propagation properties of diffracted light beams; see for example [67, Eqs. (6a,b), (7b), (10a,b)].

Our main goals are to derive a coefficients of Neumann series, to establish a closed integral representation formulae for that series and also for the modified Neumann series of the first and second kind.

The problem of computing the coefficients of the Neumann series of Bessel functions has been considered in a number of publications in the mathematical literature.

For example, Watson [130] showed that, given a function f that is analytic inside and on a circle of radius R, with center at the origin, and if C denotes the integration contour formed by that circle, then f can be expanded into a Neumann series [130, Eq. (16.1), p. 523]

$$\mathfrak{N}_0(z) = \sum_{n=0}^{\infty} \alpha_n \, J_n(z).$$

The corresponding coefficients are given by [130, Eq. (16.2), p. 523]

$$\alpha_n = \frac{\varepsilon_n}{2\pi i} \int_C f(t) \, O_n(t) dt,$$

where the functions $O_n(t)$, $n = 0, 1, \ldots$, are the Neumann polynomials, and can be obtained from

$$\frac{1}{t-z} = \sum_{n=0}^{\infty} \varepsilon_n \, O_n(t) \, J_n(z) \,,$$

where

$$\varepsilon_n = \begin{cases} 1, & \text{if } n = 0 \\ 2, & \text{if } n \in \mathbb{N} \end{cases}$$

is the so–called Neumann factor.

Wilkins [131] showed that a function $f(x)$ can be represented on $(0, \infty)$ by a Neumann series of the form

$$\mathfrak{N}_\nu^W(x) = \sum_{n=0}^{\infty} a_{n\nu} \, J_{\nu+2n+1}(x), \qquad \nu \geq -1/2 \,, \tag{3.3}$$

where the coefficients $a_{n\nu}$ are

$$\alpha_{n\nu} = 2(\nu + 2n + 1) \int_0^\infty t^{-1} f(t) \, J_{\nu+2n+1}(t) dt.$$

The problem of integral representation of Neumann series of Bessel functions occurs not so frequently. Besides the already mentioned Rice's result (3.2), there is also Wilkins who considered the possibility of integral representation for even–indexed Neumann series (3.3). Finally, let us mention Luke's integral epxression for $\mathfrak{N}_0(x)$ [61, pp.271–288] and [84, Eq. (2a)]. It is worth of mention that the *bivariate von Lommel functions of real order* are defined by even indexed Neumann–type series [130, **16.5** Eqs. (5), (6)] which ones have closed integral expressions, see [130, Eq. 16.5] and [95, Concluding remarks].

Quite recently, completely different kind integral representation for (3.1) has been given by Pogány and Süli in [95]. The main result of that article is the following theorem, which will be of great use in proving our own results:

Theorem A. (T. K. Pogány and E. Süli [95]) *Let* $\alpha \in C^1(\mathbb{R}_+)$ *and let* $\alpha|_{\mathbb{N}} = \{\alpha_n\}_{n\in\mathbb{N}}$. *Then, for all* x, ν *such that*

$$x \in \Big(0, 2\min\Big\{1, \big(e \limsup_{n\to\infty} n^{-1}|\alpha_n|^{1/n}\big)^{-1}\Big\}\Big) =: \mathcal{I}_\alpha \,, \qquad \nu > -3/2,$$

we have that

$$\mathfrak{N}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\Gamma(\nu+u+1/2)J_{\nu+u}(x)\Big) \cdot \mathfrak{d}_v\Big(\frac{\alpha(v)}{\Gamma(\nu+v+1/2)}\Big)\, du\, dv, \tag{3.4}$$

where

$$\mathfrak{d}_x := 1 + \{x\}\frac{d}{dx}\,.$$

3.1 Construction of coefficients of $\mathfrak{N}_\nu(z)$

In [95] the problem of constructing a function α, with $\alpha\big|_{\mathbb{N}} \equiv \alpha_n$, was posed, such that the integral representation (3.4) holds. The purpose of this section is to answer this open question.

The results exposed in this section concern to the paper by Jankov *et al.* [43].

We will describe the class $\Lambda = \{\alpha\}$ of functions that generate the integral representation (3.4) of the corresponding Neumann series, in the sense that the restriction $\alpha\big|_{\mathbb{N}} = (\alpha_n)$ forms the coefficient array of the series (3.1). Knowing only the set of nodes $\mathbf{N} := \big\{(n, \alpha_n)\big\}_{n\in\mathbb{N}}$ this question cannot be answered merely by examining the convergence of the series $\mathfrak{N}_\nu(x)$ and then interpolating the set $\mathbf{N}$. We formulate an answer to this question so that the resulting class of functions α depends on a suitable, integrable (on $\mathbb{R}_+$), scaling–function h.

Theorem 3.1. (D. Jankov, T. K. Pogány and E. Süli [43]) *Let* Theorem A *hold for a given convergent Neumann series of Bessel functions, and suppose that the integrand in* (3.4) *is such that*

$$\frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)J_{\nu+\omega}(x)\Big)\int_0^{[\omega]} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)d\eta \;\in\; L^1(\mathbb{R}_+)\,,$$

and let

$$h(\omega) := \frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)J_{\nu+\omega}(x)\Big)\int_0^{\omega} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)d\eta\,.$$

Then we have that

$$\alpha(\omega) = \begin{cases} \Gamma(\nu+k+1/2)\;\dfrac{d}{d\omega}\dfrac{h(\omega)}{\mathcal{B}(\omega)}\Big|_{\omega=k+}, & \text{if } \omega = k \in \mathbb{N} \\[2ex] \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}}\;\Big(\dfrac{h(\omega)}{\mathcal{B}(\omega)} - \dfrac{h(k+)}{\mathcal{B}(k)}\Big), & \text{if } 1<\omega \neq k \in \mathbb{N} \end{cases}, \tag{3.5}$$

where

$$\mathcal{B}(\omega) := \frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)\,J_{\nu+\omega}(x)\Big).$$

Proof. Assume that the integral representation (3.4) holds for some class Λ of functions α whose restriction $\alpha\big|_{\mathbb{N}}$ forms the coefficient array employed in $\mathfrak{N}_\nu(x)$. Suppose that $\widetilde{h} \in L^1(\mathbb{R}_+)$ is defined by

$$\widetilde{h}(\omega) := \frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+1/2)J_{\nu+\omega}(x)\Big)\cdot\int_0^{[\omega]} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)}\Big)d\eta; \tag{3.6}$$

in other words, $\widetilde{h}$ converges to zero sufficiently fast as $\omega \to +\infty$ so as to ensure that the integral (3.4) converges. Because $\omega \sim [\omega]$ for large ω, by (3.6) we deduce that

$$\int_0^{\omega} \partial_\eta \Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)} \Big) d\eta = \frac{h(\omega)}{\mathcal{B}(\omega)}, \tag{3.7}$$

where

$$h(\omega) = \frac{\widetilde{h}(\omega) \displaystyle\int_0^{\omega} \partial_\eta \Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)} \Big) d\eta}{\displaystyle\int_0^{[\omega]} \partial_\eta \Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+1/2)} \Big) d\eta} \sim \widetilde{h}(\omega), \qquad \omega \to \infty.$$

Differentiating (3.7) with respect to ω we get

$$\{\omega\}\alpha'(\omega) + \big(1 - \{\omega\}\psi(\nu+\omega+1/2)\big)\alpha(\omega) = \Gamma(\nu+\omega+1/2) \cdot \frac{\partial}{\partial\omega} \frac{h(\omega)}{\mathcal{B}(\omega)}, \tag{3.8}$$

where ψ denotes the familiar digamma-function, i.e. $\psi := (\ln \Gamma)'$. For integer $\omega \equiv k \in \mathbb{N}$ we know the coefficient–set $\Lambda = \{\alpha_k\}$. Therefore, let $\omega \in (k, k+1)$, where k is a fixed positive integer. By this specification (3.8) becomes a linear ODE in the unknown α:

$$\alpha'(\omega) + \Big(\frac{1}{\omega - k} - \psi(\nu+\omega+1/2) \Big) \alpha(\omega) = \frac{\Gamma(\nu+\omega+1/2)}{\omega - k} \cdot \frac{\partial}{\partial\omega} \frac{h(\omega)}{\mathcal{B}(\omega)}.$$

After some routine calculations we get

$$\alpha(\omega) = \frac{\Gamma(\nu+\omega+1/2)}{\{\omega\}} \Big(C_k + \frac{h(\omega)}{\mathcal{B}(\omega)} \Big),$$

where C_k denotes the integration constant. Thus we deduce that, for $\omega \geq 1$, we have

$$\alpha(\omega) = \begin{cases} \alpha_k, & \text{if } \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}} \Big(C_k + \dfrac{h(\omega)}{\mathcal{B}(\omega)} \Big), & \text{if } 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

It remains to find the numerical value of C_k. By the assumed convergence of $\mathfrak{N}_\nu(x), \alpha(\omega)$ has to decay to zero as $k \to \infty$. Indeed, Landau's bound [53], *viz.*

$$|J_\nu(x)| \leq c_L\, x^{-1/3}, \qquad c_L = \sup_{x \in \mathbb{R}_+} x^{1/3} J_0(x)\,,$$

clarifies this claim. Since k is not a pole of $\Gamma(\nu+\omega+1/2)$, by L'Hospital's rule we deduce that

$$\begin{aligned} \alpha_k &= \lim_{\omega \to k+} \alpha(\omega) = \lim_{\omega \to k+} \Gamma(\nu+\omega+1/2) \lim_{\omega \to k+} \frac{C_k + \dfrac{h(\omega)}{\mathcal{B}(\omega)}}{\omega - k} \\ &= \Gamma(\nu+k+1/2) \lim_{\omega \to k+} \frac{d}{d\omega} \frac{h(\omega)}{\mathcal{B}(\omega)} = \Gamma(\nu+k+1/2)\, \frac{d}{d\omega} \frac{h(\omega)}{\mathcal{B}(\omega)} \Big|_{\omega = k+}, \end{aligned}$$

such that makes sense only for

$$C_k = -\frac{h(k+)}{\mathcal{B}(k)}\,.$$

Hence

$$\alpha(\omega) = \begin{cases} \Gamma(\nu + k + 1/2)\ \dfrac{d}{d\omega}\dfrac{h(\omega)}{\mathcal{B}(\omega)}\Big|_{\omega=k+}, & \text{if } \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}}\Big(\dfrac{h(\omega)}{\mathcal{B}(\omega)} - \dfrac{h(k+)}{\mathcal{B}(k)}\Big), & \text{if } 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

This proves the assertion of the Theorem 3.1. □

3.1.1 Examples

Now, we will consider some examples of the function $\widetilde{h} \in L^1(\mathbb{R}_+)$, which describes the convergence rate to zero of the integrand in (3.6) at infinity, and $h(\omega) \sim \widetilde{h}(\omega)$, $\omega \to \infty$, where h is function from the Theorem 3.1.

Example 3.1. *Let* $\widetilde{h}(\omega) = e^{-[\omega]}$. *Since* $\int_0^\infty e^{-[\omega]}\,d\omega = e/(e-1)$, *we have that* $\widetilde{h} \in L^1(\mathbb{R}_+)$. *As* $e^{-[\omega]} \sim e^{-\omega} = h(\omega)$ *as* $\omega \to \infty$, *by* (3.5) *we conclude*

$$\alpha(\omega) = \begin{cases} \Gamma(\nu + k + 1/2)\ \dfrac{d}{d\omega}\dfrac{e^{-\omega}}{\mathcal{B}(\omega)}\Big|_{\omega=k+}, & \textit{if } \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}}\Big(\dfrac{e^{-\omega}}{\mathcal{B}(\omega)} - \dfrac{e^{-k}}{\mathcal{B}(k)}\Big), & \textit{if } 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

Example 3.2. *Let* $\widetilde{h}(\omega) = \dfrac{[\omega]^{\beta-1}}{e^{[\omega]}-1}$, $\beta > 1$; *then*

$$\int_0^\infty \widetilde{h}(\omega)\,d\omega = \sum_{n=1}^\infty \frac{(n-1)^{\beta-1}}{e^{n-1}-1},$$

which is a convergent series, so $\widetilde{h} \in L^1(\mathbb{R}_+)$. *As* $\omega \to \infty$ *we have that*

$$[\omega]^{\beta-1}\big(e^{[\omega]}-1\big)^{-1} \sim \omega^{\beta-1}\big(e^{\omega}-1\big)^{-1} = h(\omega).$$

Hence $\int_0^\infty h(\omega)\,d\omega = \Gamma(\beta)\zeta(\beta)$, *where* ζ *is Riemann's* ζ *function. Then, for such* β, (3.5) *gives*

$$\alpha(\omega) = \begin{cases} \Gamma(\nu + k + 1/2)\ \dfrac{d}{d\omega}\dfrac{\omega^{\beta-1}}{(e^\omega-1)\mathcal{B}(\omega)}\Big|_{\omega=k+}, & \textit{if } \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}}\Big(\dfrac{\omega^{\beta-1}}{\mathcal{B}(\omega)\,(e^\omega-1)} \\ \quad -\dfrac{k^{\beta-1}}{\mathcal{B}(k)\,(e^k-1)}\Big), & \textit{if } 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

Example 3.3. *Let* $\widetilde{h}(\omega) = e^{-s[\omega]}J_0([\omega])$, *where* $s > 1$ *and* J_0 *is the Bessel function of the first kind of order zero. Since*

$$\int_0^\infty e^{-s[\omega]}J_0([\omega])\,d\omega = \sum_{n=1}^\infty e^{-s(n-1)}J_0(n-1),$$

we see that $\widetilde{h} \in L^1(\mathbb{R}_+)$. *Because* $e^{-s[\omega]}J_0([\omega]) \sim e^{-s\omega}J_0(\omega) = h(\omega)$ *as* $\omega \to \infty$, *and* $\int_0^\infty h(\omega)\,d\omega = (s^2+1)^{-1/2}$, *from* (3.5) *we deduce*

$$\alpha(\omega) = \begin{cases} \Gamma(\nu+k+1/2)\,\dfrac{d}{d\omega}\dfrac{e^{-s\omega}J_0(\omega)}{\mathcal{B}(\omega)}\Bigg|_{\omega=k+}, & \text{if } \omega = k \in \mathbb{N} \\[2ex] \dfrac{\Gamma(\nu+\omega+1/2)}{\{\omega\}}\Big(\dfrac{e^{-s\omega}J_0(\omega)}{\mathcal{B}(\omega)} - \dfrac{e^{-sk}J_0(k)}{\mathcal{B}(k)}\Big), & \text{if } 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

3.2 Integral representations for $\mathfrak{N}_\nu(x)$ *via* Bessel differential equation

Previously, we introduced an integral representation (3.4) of Neumann series (3.1), compare Theorem A. The purpose of this section is to establish another (indefinite) integral representations for Neumann series of Bessel function by means of Chessin's results [13, 14] and by applying the variation of parameters method. Finally, using fractional differintegral approach in solving the nonhomogeneous Bessel ordinary differential equation [58, 59, 127, 128, 129] we derive integral expression formulæ for $\mathfrak{N}_\nu(x)$.

The listed results are taken from the paper Baricz *et al.* [6].

3.2.1 Approach by Chessin

One of the crucial arguments used in the proof of our main results is the simple fact that the Bessel functions of the first kind are actually particular solutions of the second-degree homogeneous Bessel differential equation. We note that this approach in the study of the Neumann series of Bessel functions is much simpler than the previous methods which we have found in the literature. In the geometric theory of univalent functions the idea to use Bessel's differential equation is also useful in the study of geometric properties (like univalence, convexity, starlikeness, close-to-convexity) of Bessel functions of the first kind. For more details we refer to the monograph [3].

In the sequel we shall need the *Bessel functions of the second kind of order* ν (or MacDonald functions) $Y_\nu(x)$ which satisfy

$$Y_\nu(x) = \operatorname{cosec}(\pi\nu)\big(J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)\big), \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi\,, \tag{3.9}$$

and which have the following differentiability properties

$$W[J_\nu, Y_\nu](z) = \frac{2}{\pi z}, \quad W[J_{-\nu}, J_\nu](z) = \frac{2\sin(\nu\pi)}{\pi z}, \qquad \nu \in \mathbb{R}, z \neq 0\,, \tag{3.10}$$

valid for the related Wronskians $W[\cdot,\cdot](x)$.

Explicit solution of Bessel differential equation with general nonhomogeneous part

$$y'' + \frac{1}{x}y' + \Big(1 - \frac{\nu^2}{x^2}\Big)y = f(x)\,, \tag{3.11}$$

has been derived for the first time in a set of articles by Chessin more than a century ago, see for example [13, 14]. In [13, p. 678] Chessin differs the cases:

- For $\nu = n \in \mathbb{Z}$ the solution is given by

$$y(x) = A(x)J_n(x) + B(x)Y_n(x), \tag{3.12}$$

and

$$A'(x) = \frac{Y_n(x)f(x)}{W[Y_n, J_n](x)} = -\frac{\pi x Y_n(x)f(x)}{2}, \quad B'(x) = -\frac{J_n(x)f(x)}{W[Y_n, J_n](x)} = \frac{\pi x J_n(x)f(x)}{2}\,.$$

- If $\nu \notin \mathbb{Z}$, we have

$$y(x) = A_1(x)J_\nu(x) + B_1(x)J_{-\nu}(x), \tag{3.13}$$

where

$$A_1'(x) = \frac{J_{-\nu}(x)f(x)}{W[J_{-\nu}, J_\nu](x)} = \frac{\pi x J_{-\nu}(x)f(x)}{2\sin(\nu\pi)}, \quad B_1'(x) = -\frac{J_\nu(x)f(x)}{W[J_{-\nu}, J_\nu](x)} = -\frac{\pi x J_\nu(x)f(x)}{2\sin(\nu\pi)}\,.$$

Consider the homogeneous Bessel differential equation of $(n+\nu)$-th index

$$x^2y'' + xy' + (x^2 - (n+\nu)^2)y = 0\,, \qquad n \in \mathbb{N}\,,\, 2\nu+3 > 0\,,$$

which particular solution is $J_{n+\nu}(x)$, that is

$$x^2J''_{n+\nu}(x) + xJ'_{n+\nu}(x) + (x^2 - (n+\nu)^2)J_{n+\nu}(x) = 0\,. \tag{3.14}$$

Multiplying (3.14) by α_n, then summing up this expression with respect to $n \in \mathbb{N}$ we arrive at

$$x^2\mathfrak{N}''_\nu(x) + x\mathfrak{N}'_\nu(x) + (x^2 - \nu^2)\mathfrak{N}_\nu(x) = \mathfrak{P}_\nu(x) := \sum_{n=1}^{\infty} n(n+2\nu)\alpha_n J_{n+\nu}(x); \tag{3.15}$$

the right side expression $\mathfrak{P}_\nu(x)$ defines the so–called *Neumann series of Bessel functions associated to* $\mathfrak{N}_\nu(x)$. Obviously (3.15) turns out to be a nonhomogeneous Bessel differential equation in unknown function $\mathfrak{N}_\nu(x)$, while by virtue of substitution $\alpha_n \mapsto n(n+2\nu)\alpha_n$, Theorem A gives

$$\mathfrak{P}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\Gamma\big(\nu+u+1/2\big)J_{\nu+u}(x)\Big) \cdot \partial_s\Big(\frac{s\,(s+2\nu)\,\alpha(s)}{\Gamma(\nu+s+1/2)}\Big)\,du\,ds. \tag{3.16}$$

Let us find the domain of associated Neumann series $\mathfrak{P}_\nu(x)$. Theorem A gives the same range of validity $x \in \mathcal{I}_\alpha$ by means of the estimate

$$\big|\mathfrak{P}_\nu(x)\big| \le \sum_{n=1}^{\infty} n(n+2\nu)|\alpha_n|\,\big|J_{n+\nu}(x)\big|\,.$$

since $\limsup_{n\to\infty}\{n(n+2\nu)\}^{1/n} = 1$.

Using the Landau's bound (2.22) we see that $\mathfrak{P}_\nu(x)$ is defined for all $x \in \mathcal{I}_\alpha$ when series $\sum_{n=1}^\infty n^{5/3}\alpha_n$ absolutely converges such that clearly follows from

$$\left|\mathfrak{P}_\nu(x)\right| \leq b_L \sum_{n=1}^\infty \frac{n\,(n+2\nu)}{(n+\nu)^{1/3}}|\alpha_n|\,.$$

Now, we are ready to formulate our first main result in this section.

Theorem 3.2. (Á. Baricz, D. Jankov and T. K. Pogány [6]) *Let* $\alpha \in C^1(\mathbb{R}_+)$ *and let* $\alpha|_{\mathbb{N}} = \{\alpha_n\}_{n\in\mathbb{N}}$ *and assume that* $\sum_{n=1}^\infty n^{5/3}\alpha_n$ *absolutely converges. Then for all* $x \in \mathcal{I}_\alpha, \nu > -1/2$ *we have*

$$\mathfrak{N}_\nu(x) = \begin{cases} \dfrac{\pi}{2}\left(Y_n(x)\displaystyle\int \frac{J_n(x)\mathfrak{P}_\nu(x)}{x}dx - J_n(x)\int \frac{Y_n(x)\mathfrak{P}_\nu(x)}{x}dx\right), & \text{if } \nu = n \in \mathbb{Z} \\ \dfrac{\pi}{2\sin(\nu\pi)}\left(J_\nu(x)\displaystyle\int \frac{J_{-\nu}(x)\mathfrak{P}_\nu(x)}{x}dx \right. & \\ \qquad\qquad \left. -J_{-\nu}(x)\displaystyle\int \frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x}dx\right), & \text{if } \nu \notin \mathbb{Z} \end{cases}. \tag{3.17}$$

Proof. It is enough to substitute $f(x) \equiv x^{-2}\mathfrak{P}_\nu(x)$ in nonhomogeneous Bessel differential equation (3.11) and calculate integrals in (3.12) and (3.13), using into account the differentiability properties (3.10). Then, by Chessin's procedure we arrive at the asserted expressions (3.17). □

Remark 3.3. Chessin's derivation procedure is in fact the variation of parameters method. Repeating calculations by variation of parameters method we will arrive at

$$\mathfrak{N}_\nu(x) = \frac{\pi}{2}\left(Y_\nu(x)\int \frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x}\,dx - J_\nu(x)\int \frac{Y_\nu(x)\mathfrak{P}_\nu(x)}{x}\,dx\right),$$

where $\nu > -1/2$, $x \in \mathcal{I}_\alpha$. ■

Theorem 3.4. (Á. Baricz, D. Jankov and T. K. Pogány [6]) *Let the situation be the same as in* Theorem 3.2. *Then for* $\sum_{n=1}^\infty n^{5/3}|\alpha_n| < \infty$, *we have*

$$\begin{aligned}\mathfrak{N}_\nu(x) &= \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{\mathfrak{P}_\nu(x)\cdot J_\nu(x)}{x}\,dx\right)dx \\ &\quad + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{\mathfrak{P}_\nu(x)\cdot Y_\nu(x)}{x}\,dx\right)dx\,,\end{aligned}$$

where $\mathfrak{P}_\nu$ *stands for the Neumann series* (3.16) *associated with the initial Neumann series of Bessel functions* $\mathfrak{N}_\nu(x), x \in \mathcal{I}_\alpha$.

Proof. We apply now the reduction of order method in solving the Bessel equation. Solution of

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0 \tag{3.18}$$

in $\mathcal{I}_\alpha$ is given by

$$y_h(x) = C_1Y_\nu(x) + C_2J_\nu(x)\,.$$

It is well known that J_ν and Y_ν are independent solutions of the homogeneous Bessel differential equation (3.18), since the Wronskian $W(x) = W[J_\nu(x), Y_\nu(x)] = 2/(\pi x) \neq 0$, $x \in \mathcal{I}_\alpha$.

Since $J_\nu(x)$ is a solution to the homogeneous ordinary differential equation, a guess of the particular solution is $\mathfrak{N}_\nu(x) = J_\nu(x)w(x)$. Substituting this form into (3.18) we get

$$x^2(J_\nu''w + 2J_\nu'w' + J_\nu w'') + x(J_\nu'w + J_\nu w') + (x^2 - \nu^2)J_\nu w = \mathfrak{P}_\nu(x).$$

Rewriting the equation as

$$w\big(x^2J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu\big) + w'(2x^2J_\nu' + xJ_\nu) + w''\, x^2J_\nu = \mathfrak{P}_\nu(x),$$

shows that the first term vanishes being J_ν solution of the homogeneous part of (3.18). This leaves the following linear ordinary differential equation for w':

$$(w')' + \frac{2xJ_\nu' + J_\nu}{xJ_\nu}\, w' = \frac{\mathfrak{P}_\nu(x)}{x^2J_\nu}\,.$$

Hence

$$w' = \frac{1}{xJ_\nu^2}\int \frac{\mathfrak{P}_\nu \cdot J_\nu}{x}\, dx + \frac{C_3}{xJ_\nu^2}\,,$$

i.e.

$$w = \int \frac{1}{xJ_\nu^2}\Big\{\int \frac{\mathfrak{P}_\nu \cdot J_\nu}{x}\, dx\Big\}\, dx + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}\, dx = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}\,.$$

Being J_ν, Y_ν independent, that make up the homogeneous solution, they do not contribute to the particular solution and the constants C_3, C_4 can be set to be zero.

Now, we can take particular solution in the form $\mathfrak{N}_\nu(x) = Y_\nu(x)w(x)$, and analogously as above, we get

$$\mathfrak{N}_\nu(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{B}_\nu \cdot Y_\nu}{x}\, dx\right) dx - C_5\frac{\pi}{2}J_\nu(x) + C_6Y_\nu(x)\,,$$

having in mind that

$$\int \frac{1}{xY_\nu^2}\, dx = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}.$$

Choosing $C_5 = C_6 = 0$, we complete the proof of the asserted result. □

3.2.2 Solutions of Bessel differential equation using a fractional differintegral

In this section we will give the solution of nonhomogeneous Bessel differential equation, using properties associated with the fractional differintegration which was introduced in Chapter 2.

Below, we shall need the result given as the part of e.g. [58, p. 1492, Theorem 3], [128, p. 109, Theorem 3]). We recall the mentioned result in our setting. Thus, if $\mathfrak{P}_\nu(x), x \in \mathcal{I}_\alpha$ satisfies the constraint (3.15) and $\big(\mathfrak{P}_\nu(x)\big)_{-\mu} \neq 0$, then the following nonhomogeneous linear ordinary differential equation (3.15) has a particular solution $y_p = y_p(x)$ in the form

$$y_p(x) = x^\mu e^{\lambda x}\left(\left(x^{\mu-1/2}e^{2\lambda x}\left(x^{-\mu-1}e^{-\lambda x}\mathfrak{P}_\nu(x)\right)_{-\mu-1/2}\right)_{-1} x^{-\mu-1/2}e^{-2\lambda x}\right)_{\mu-1/2} \tag{3.19}$$

where $\mu \in \mathbb{R}$; $\lambda = \pm i$; $x \in (\mathbb{C}\setminus\mathbb{R})\cup\mathcal{I}_\alpha$, provided that $\mathfrak{P}_\nu(x)$. Let us simplify (3.19), using the generalized Leibniz rule [58, p. 1489, Lemma 3]:

$$\begin{aligned}\left(x^{-\mu-1}e^{-\lambda x}\mathfrak{P}_\nu(x)\right)_{-\mu-1/2} &= \sum_{n=0}^\infty \binom{-\mu-1/2}{n}(x^{-\mu-1}e^{-\lambda x})_{-\mu-1/2-n}\left(\mathfrak{P}_\nu(x)\right)_n\\
&= \sum_{n,k=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}(x^{-\mu-1})_{-\mu-1/2-n-k}(e^{-\lambda x})_k\left(\mathfrak{P}_\nu(x)\right)_n\\
&= \frac{\Lambda_\mu(x)}{\pi}\sum_{n,k=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}(-x)^n(\lambda x)^k\Gamma(-n-k+1/2)\left(\mathfrak{P}_\nu(x)\right)_n,\end{aligned}$$

where

$$\Lambda_\mu(x) = \frac{\pi e^{i\pi(\mu+1/2)-\lambda x}}{\Gamma(\mu+1)\sqrt{x}}\,.$$

By Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi\mathrm{cosec}(\pi z)$, we get

$$\left(x^{-\mu-1}e^{-\lambda x}\mathfrak{P}_\nu(x)\right)_{-\mu-1/2} = \Lambda_\mu(x)\sum_{n,k=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}\frac{x^n(-\lambda x)^k\left(\mathfrak{P}_\nu(x)\right)_n}{\Gamma(1/2+n+k)}\,.$$

Now we have

$$\begin{aligned}&\left(x^{\mu-1/2}e^{2\lambda x}\left(x^{-\mu-1}e^{-\lambda x}\mathfrak{P}_\nu(x)\right)_{-\mu-1/2}\right)_{-1}\\
&\quad= \frac{\pi e^{i\pi(\mu+1/2)}}{\Gamma(\mu+1)}\sum_{n,k=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}\frac{(-\lambda)^k\left(x^{\mu+n+k-1}e^{\lambda x}\left(\mathfrak{P}_\nu(x)\right)_n\right)_{-1}}{\Gamma(1/2+n+k)}\\
&\quad= \frac{\pi x^\mu\, e^{i\pi(\mu+1/2)+\lambda x}}{\Gamma(\mu+1)}\sum_{n,k=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}\frac{(-\lambda)^k x^{n+k}\left(\mathfrak{P}_\nu(x)\right)_n}{(\mu+n+k)\Gamma(1/2+n+k)}\,.\end{aligned}$$

Finally, after some simplification (again by Euler's reflection formula) we get

$$\begin{aligned}y_p(x) = \frac{1}{\Gamma(\mu+1)}&\sum_{n,k,\ell,m=0}^\infty \binom{-\mu-1/2}{n}\binom{-\mu-1/2-n}{k}\binom{-\mu-1/2}{\ell}\binom{-\mu-1/2-\ell}{m}\\
&\cdot\frac{(-1)^{1+\ell+n}\lambda^{k+m}\Gamma(\mu-\ell-m-n-k)}{(\mu+n+k)}x^{n+k+\ell+m}\left(\mathfrak{P}_\nu(x)\right)_{n+\ell}\,.\end{aligned} \tag{3.20}$$

These in turn imply the following result.

Theorem 3.5. (Á. Baricz, D. Jankov and T. K. Pogány [6]) *Let* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_\mathbb{N} = \{\alpha_n\}_{n\in\mathbb{N}}$ *and assume that* $\sum_{n=1}^\infty n^{5/3}\alpha_n$ *absolutely converges. Then for all* $x \in (\mathbb{C}\setminus\mathbb{R})\cup\mathcal{I}_\alpha, \nu > -1/2$ *there holds*

$$\mathfrak{N}_\nu(x) = y_p(x)\,,$$

where y_p *is given by* (3.20).

Remark 3.6. In [58, p. 1492, Theorem 3] it is given solution of the homogeneous differential equation

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0$$

in the form

$$y_h(x) = Kx^{\mu}e^{\lambda x}\left(x^{-\mu-1/2}e^{-2\lambda x}\right)_{\mu-1/2} \tag{3.21}$$

for all $\mu \in \mathbb{R}$; $\lambda = \pm i$; $x \in (\mathbb{C}\setminus\mathbb{R}) \cup \mathcal{I}_\alpha$ and where K is an arbitrary real constant. Then, summing (3.20) and (3.21) we can get another solution of nonhomogeneous linear ordinary differential equation (3.15). ■

3.2.3 Fractional integral representation

Recently Lin, Srivastava and coworkers devoted articles to explicit fractional solutions of nonhomogeneous Bessel differential equation, such that turn out to be a special case of the Tricomi equation [58, 59, 128, 129]. In this section we will exploite their results to obtain further integral representation formulae for the Neumann series $\mathfrak{N}_\nu(x)$.

Using the fractional–calculus approach the mentioned we have obtain the following solutions of the homogeneous Bessel differential equation, depending on the parameter ν which can be found in [127]:

- For $\nu = n + 1/2$, $n \in \mathbb{N}_0$, the solution is given by

  $$y_h(x) = K_1 J_{-n-1/2}(x) + K_2 J_{n+1/2}(x),$$

  where K_1 and K_2 are arbitrary constants, and

  $$\begin{aligned} J_{-n-1/2}(x) &= \sqrt{\frac{2}{\pi x}}\left(\cos\left(x + \frac{\pi}{2}n\right)\sum_{k=0}^{[n/2]}(-1)^k\frac{(n+2k)!}{(2k)!(n-2k)!}(2x)^{-2k}\right.\\ &\left.\quad - \sin\left(x+\frac{\pi}{2}n\right)\sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{(n+2k+1)!}{(2k+1)!(n-2k-1)!}(2x)^{-2k-1}\right), \end{aligned} \tag{3.22}$$

  $$\begin{aligned} J_{n+1/2}(x) &= \sqrt{\frac{2}{\pi x}}\left(\sin\left(x - \frac{\pi}{2}n\right)\sum_{k=0}^{[n/2]}(-1)^k\frac{(n+2k)!}{(2k)!(n-2k)!}(2x)^{-2k}\right.\\ &\left.\quad + \cos\left(x-\frac{\pi}{2}n\right)\sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{(n+2k+1)!}{(2k+1)!(n-2k-1)!}(2x)^{-2k-1}\right). \end{aligned} \tag{3.23}$$

- For $\nu \notin \mathbb{Z}$ the solution is

  $$y_h(x) = K_1 J_{-\nu}(x) + K_2 J_\nu(x),$$

where K_1 and K_2 are arbitrary constants, and asymptotic estimates for $J_{-\nu}$ and J_ν follows from equations (3.22) and (3.23), respectively, i.e.

$$\begin{aligned} J_{-\nu}(x) \sim & \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2x)^{-2k} \\ & - \sin\left(x+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2x)^{-2k-1}\Bigg), \\ J_{\nu}(x) \sim & \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2x)^{-2k} \\ & - \sin\left(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2x)^{-2k-1}\Bigg) \end{aligned}$$

each of which is valid for large values of $|x|$ provided that $|\arg(x)| \le \pi - \epsilon$, $(0 < \epsilon < \pi)$.

(iii) In the case when $\nu = n \in \mathbb{Z}$, two linearly independent solutions which make a general solution of Bessel differential equation, are J_n and

$$\begin{aligned} Y_n(x) \underset{n\to\infty}{\sim} & \sqrt{\frac{2}{\pi x}}\Bigg(\sin\left(x-\frac{\pi}{2}n-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+1/2)}{(2k)!\Gamma(n-2k+1/2)!}(2x)^{-2k} \\ & + \cos\left(x-\frac{\pi}{2}n-\frac{\pi}{4}\right)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+2k+3/2)}{(2k+1)!\Gamma(n-2k-1/2)}(2x)^{-2k-1}\Bigg). \end{aligned}$$

Using the previous results we easily get the following results.

Theorem 3.7. (Á. Baricz, D. Jankov and T. K. Pogány [6]) *Let the conditions from* Theorem 3.2 *hold. Then, the integral representation formulae for the function* $\mathfrak{N}_\nu(x)$ *reads as follows:*

- *for* $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}_0$, *we have*

$$\mathfrak{N}_{n+1/2}(x) = \frac{(-1)^n\pi}{2}\left(J_{n+1/2}(x)\int\frac{J_{-n-1/2}(x)\mathfrak{P}_\nu(x)}{x}\,dx - J_{-n-1/2}(x)\int\frac{J_{n+1/2}(x)\mathfrak{P}_\nu(x)}{x}\,dx\right); \tag{3.24}$$

- *for* $\nu \notin \mathbb{Z}$, *it is*

$$\mathfrak{N}_\nu(x) = \frac{\pi}{2\sin(\nu\pi)}\left(J_\nu(x)\int\frac{J_{-\nu}(x)\mathfrak{P}_\nu(x)}{x}\,dx - J_{-\nu}(x)\int\frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x}\,dx\right). \tag{3.25}$$

Here $J_{\mp n\mp 1/2}(x)$ *are given in* (3.22) *and* (3.23) *respectively and* $\mathfrak{P}_\nu$ *stands for the Neumann series* (3.16) *associated with the initial Neumann series* $\mathfrak{N}_\nu(x), x \in \mathcal{I}_\alpha$.

Proof. By variation of parameters method and by virtue of (3.10) we get the representations (3.24) and (3.25). □

3.3 Integral representations for Neumann–type series of Bessel functions

In this section we cite the results from the paper Baricz *et al.* [5].

Here we pose the problem of integral representation for another Neumann–type series of Bessel functions when J_ν is replaced in (3.1) by modified Bessel function of the first kind I_ν, Bessel functions of the second kind Y_ν, K_ν (called Basset–Neumann and MacDonald functions respectively), Hankel functions $H_\nu^{(1)}, H_\nu^{(2)}$ (or Bessel functions of the third kind), of which precise definitions can be found in [130].

According to the established nomenclatures in the sequel we will distinguish Neumann series of first, second and third kind depending on the Bessel functions which build this series. So, the *first kind Neumann series* are

$$\mathfrak{N}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n\, J_{\nu+n}(z), \qquad \mathfrak{M}_\nu(z) := \sum_{n=1}^{\infty} \beta_n\, I_{\nu+n}(z)\,. \tag{3.26}$$

The *second kind Neumann series* we introduce as

$$\mathfrak{J}_\nu(z) := \sum_{n=1}^{\infty} \delta_n\, K_{\nu+n}(z), \qquad \mathfrak{X}_\nu(z) := \sum_{n=1}^{\infty} \gamma_n\, Y_{\nu+n}(z)\,. \tag{3.27}$$

In the next two sections our aim is to present closed form expressions for these Neumann series occurring in (3.26) and (3.27). Our main tools include Laplace–integral form of a Dirichlet series [48], the condensed form of Euler–Maclaurin summation formula [95, p. 2365] and certain bounding inequalities for I_ν and K_ν, see [4].

3.3.1 Integral representation for the first kind Neumann series $\mathfrak{M}_\nu(x)$

First, we present an integral representation for the first kind Neumann series $\mathfrak{M}_\nu(x)$, where I_ν is the modified Bessel function of the first kind of order ν, defined by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)\,n!}, \qquad z,\nu \in \mathbb{C}\,.$$

Theorem 3.8. (Á. Baricz, D. Jankov and T.K. Pogány [5]) *Let* $\beta \in C^1(\mathbb{R}_+)$, $\beta|_{\mathbb{N}} = \{\beta_n\}_{n\in\mathbb{N}}$ *and assume that* $\sum\limits_{n=1}^{\infty} \beta_n$ *is absolutely convergent. Then, for all*

$$x \in \left(0, 2\min\left\{1, \left(e\, \limsup_{n\to\infty} n^{-1}|\beta_n|^{1/n}\right)^{-1}\right\}\right) =: \mathcal{I}_\beta, \qquad \nu > -3/2,$$

we have the integral representation

$$\mathfrak{M}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\Gamma(\nu+u+1/2)\, I_{\nu+u}(x)\right) \cdot \partial_s\left(\frac{\beta(s)}{\Gamma(\nu+s+1/2)}\right) du\, ds\,.$$

Proof. First, we establish the convergence conditions of the first kind Neumann series $\mathfrak{M}_\nu(x)$. By virtue of the bounding inequality [4, p. 583]:

$$I_\mu(x) < \frac{\left(\frac{x}{2}\right)^\mu}{\Gamma(\mu+1)}\, e^{\frac{x^2}{4(\mu+1)}}, \qquad x>0, \mu+1>0,$$

and having in mind that $\mathcal{I}_\beta \subseteq (0,2)$, we conclude that

$$|\mathfrak{M}_\nu(x)| < \max_{n\in\mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)}\, e^{\frac{x^2}{4(\nu+n+1)}} \sum_{n=1}^\infty |\beta_n| = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)}\, e^{\frac{x^2}{4(\nu+2)}} \sum_{n=1}^\infty |\beta_n|,$$

so, the absolute convergence of $\sum\limits_{n=1}^\infty \beta_n$ suffices for the finiteness of $\mathfrak{M}_\nu(x)$ on $\mathcal{I}_\beta$. Here we used tacitly that for $x \in \mathcal{I}_\beta$ and $\nu > -1$ fixed, the function

$$\alpha \mapsto f(\alpha) = \frac{\left(\frac{x}{2}\right)^{\nu+\alpha}}{\Gamma(\nu+\alpha+1)}\, e^{\frac{x^2}{4(\nu+\alpha+1)}}$$

is decreasing on $[\alpha_0, \infty)$, where $\alpha_0 \approx 1.4616$ denotes the abscissa of the minimum of Γ, because Γ is increasing on $[\alpha_0,\infty)$ and then

$$\frac{f'(\alpha)}{f(\alpha)} = \ln\left(\frac{x}{2}\right) - \frac{x^2}{4(\nu+\alpha+1)^2} - \frac{\Gamma'(\nu+\alpha+1)}{\Gamma(\nu+\alpha+1)} \le 0.$$

Consequently, for all $n \in \{2,3,\dots\}$ we have $f(n) \le f(2)$. Moreover, by using the inequality $e^x \ge 1+x$, it can be shown easily that $f(1) \ge f(2)$ for all $x>0$ and $\nu>-1$. These in turn imply that indeed $\max\limits_{n\in\mathbb{N}} f(n) = f(1)$, i.e.

$$\max_{n\in\mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)}\, e^{\frac{x^2}{4(\nu+n+1)}} = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)}\, e^{\frac{x^2}{4(\nu+2)}},$$

as we required.

Now, recall the following integral representation [130, p. 79]

$$I_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \cosh(zt)\mathrm{d}t, \qquad z\in\mathbb{C},\ \Re(\nu) > -1/2, \tag{3.28}$$

which will be used in the sequel. Since (3.28) is valid only for $\nu > -1/2$, in what follows for the Neumann series $\mathfrak{M}_\nu(x)$ we suppose that $\nu > -3/2$. Setting (3.28) into right-hand series in (3.26) we have

$$\mathfrak{M}_\nu(x) = \sqrt{\frac{2x}{\pi}} \int_0^1 \cosh(xt) \left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2} \mathcal{D}_\beta(t)\mathrm{d}t, \qquad x>0, \tag{3.29}$$

with the Dirichlet series

$$\mathcal{D}_\beta(t) := \sum_{n=1}^\infty \frac{\beta_n}{\Gamma(n+\nu+1/2)} \exp\left(-n \ln \frac{2}{x(1-t^2)}\right). \tag{3.30}$$

Following the lines of the proof of [95, Theorem] we deduce that the x–domain is

$$0 < x < 2\min\left\{1, \left(e \limsup_{n\to\infty} n^{-1} \sqrt[n]{|\beta_n|}\right)^{-1}\right\}.$$

For such x, the convergent Dirichlet series (3.30) possesses a Laplace–integral form

$$\mathcal{D}_\beta(t) = \ln\frac{2}{x(1-t^2)} \int_0^\infty \left(\frac{x(1-t^2)}{2}\right)^u \left(\sum_{j=1}^{[u]} \frac{\beta_j}{\Gamma(j+\nu+1/2)}\right) du. \tag{3.31}$$

Expressing (3.31) *via* the condensed Euler–Maclaurin summation formula (2.12), we get

$$\mathcal{D}_\beta(t) = \ln\frac{2}{x(1-t^2)} \int_0^\infty \int_0^{[u]} \left(\frac{x(1-t^2)}{2}\right)^u \cdot \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+1/2)}\right) du\, ds. \tag{3.32}$$

Substituting (3.32) into (3.29) we get

$$\begin{aligned}\mathfrak{M}_\nu(x) = &-\sqrt{\frac{2x}{\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+1/2)}\right)\\ &\cdot\left(\int_0^1 \cosh(xt)\left(\frac{x(1-t^2)}{2}\right)^{\nu+u-1/2} \ln\frac{x(1-t^2)}{2} dt\right) du\, ds.\end{aligned} \tag{3.33}$$

Now, let us simplify the t–integral in (3.33)

$$\mathcal{J}_x(w) := \int_0^1 \cosh(xt)\cdot\left(\frac{x(1-t^2)}{2}\right)^w \ln\frac{x(1-t^2)}{2}\, dt, \qquad w := \nu+u-1/2. \tag{3.34}$$

Indefinite integration under the sign of integral in (3.34) results in

$$\int \mathcal{J}_x(w) dw = \left(\frac{x}{2}\right)^w \int_0^1 \cosh(xt)(1-t^2)^w dt = \sqrt{\frac{\pi}{2x}}\,\Gamma(w+1) I_{w+1/2}(x).$$

Now, observing that $dw = du$, we get

$$\mathcal{J}_x(\nu+u-1/2) = \sqrt{\frac{\pi}{2x}}\frac{\partial}{\partial u}\left(\Gamma(\nu+u+1/2)\, I_{\nu+u}(x)\right).$$

From (3.33) and (3.34), we immediately get the proof of the theorem, with the assertion that the integration domain $\mathbb{R}_+$ changes to $[1,\infty)$ because $[u]$ is equal to zero for all $u \in [0,1)$. □

3.3.2 Integral representation for the second kind Neumann series $\mathfrak{J}_\nu(x)$ and $\mathfrak{X}_\nu(x)$

Below, we present an integral representation for the Neumann–type series $\mathfrak{J}_\nu(x)$.

Theorem 3.9. (Á. Baricz, D. Jankov and T.K. Pogány [5]) *Let* $\delta \in C^1(\mathbb{R}_+)$ *and let* $\delta|_{\mathbb{N}} = \{\delta_n\}_{n\in\mathbb{N}}$. *Then for all* $\nu > 0$ *and*

$$x \in \mathcal{I}_\delta := \left(\frac{2}{e}\limsup_{n\to\infty}\, n|\delta_n|^{1/n}, +\infty\right),$$

we have the integral representation

$$\mathfrak{J}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} K_{\nu+u}(x) \cdot \partial_s \delta(s)\, du\, ds\,.$$

Proof. We begin by establishing first the convergence conditions for $\mathfrak{J}_\nu(x)$. To this aim let us consider the integral representation referred to Basset [130, p. 172]:

$$K_\nu(x) = \frac{2^\nu\, \Gamma(\nu+1/2)}{x^\nu \sqrt{\pi}} \int_0^\infty \frac{\cos(xt)}{(1+t^2)^{\nu+1/2}}\, dt, \qquad \Re(\nu) > -1/2,\, \Re(x) > 0\,.$$

Consequently, for all $\Re(\nu) > 0$, $x > 0$ there holds

$$K_\nu(x) \leq \frac{2^\nu\, \Gamma(\nu+1/2)}{x^\nu \sqrt{\pi}} \int_0^\infty \frac{dt}{(1+t^2)^{\nu+1/2}} = \frac{1}{2}\left(\frac{2}{x}\right)^\nu \Gamma(\nu)\,. \tag{3.35}$$

Now, recalling that $\Gamma(s) = \sqrt{2\pi}\, s^{s-1/2} e^{-s}\left(1 + \mathcal{O}(s^{-1})\right)$, $|s| \to \infty$, we have

$$|\mathfrak{J}_\nu(x)| \leq \frac{1}{2}\left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty |\delta_n| \Gamma(\nu+n) \left(\frac{2}{x}\right)^n \sim \sqrt{\frac{\pi}{2}} \left(\frac{2}{ex}\right)^\nu \sum_{n=1}^\infty (\nu+n)^{\nu+n-1/2} |\delta_n| \left(\frac{2}{ex}\right)^n$$

where the last series converges uniformly for all $\nu > 0$ and $x \in \mathcal{I}_\delta$. Note that more convenient integral representation for the modified Bessel function of the second kind is [130, p. 183]

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1} e^{-t-\frac{x^2}{4t}}\, dt, \qquad |\arg(x)| < \pi/2,\, \Re(\nu) > 0. \tag{3.36}$$

Thus, combining the right-hand equality in (3.27) and (3.36) we get

$$\mathfrak{J}_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1} e^{-t-\frac{x^2}{4t}} \cdot \mathcal{D}_\delta(t)\, dt, \qquad x \in \mathcal{I}_\delta, \tag{3.37}$$

where $\mathcal{D}_\delta(t)$ is the Dirichlet series

$$\mathcal{D}_\delta(t) = \sum_{n=1}^\infty \delta_n \left(\frac{x}{2t}\right)^n = \sum_{n=1}^\infty \delta_n \exp\left(-n \ln \frac{2t}{x}\right). \tag{3.38}$$

The Dirichlet series' parameter is necessarily positive, therefore (3.38) converges for all $x \in \mathcal{I}_\delta$. Now, the related Laplace–integral and the Euler–Maclaurin summation formula give us:

$$\mathcal{D}_\delta(t) = \ln\frac{2t}{x} \int_0^\infty \int_0^{[u]} \left(\frac{x}{2t}\right)^u \cdot \partial_s \delta(s)\, du\, ds. \tag{3.39}$$

Substituting (3.39) into (3.37) we get

$$\mathfrak{J}_\nu(x) = -\frac{x^\nu}{2^{\nu+1}} \int_0^\infty \int_0^{[u]} \partial_s \delta(s) \left(\int_0^\infty \left(\frac{x}{2t}\right)^u \ln\left(\frac{x}{2t}\right) t^{-\nu-1} e^{-t-\frac{x^2}{4t}} dt\right) du\, ds. \tag{3.40}$$

Denoting

$$\mathcal{I}_x(u) := \int_0^\infty \left(\frac{x}{2t}\right)^u \ln\left(\frac{x}{2t}\right) t^{-\nu-1} e^{-t-\frac{x^2}{4t}} dt,$$

we obtain
$$\int \mathcal{I}_x(u)du = \left(\frac{x}{2}\right)^u \int_0^\infty t^{-(\nu+u)-1} e^{-t-\frac{x^2}{4t}} dt = 2\left(\frac{2}{x}\right)^\nu K_{\nu+u}(x)\,.$$
Therefore
$$\mathcal{I}_x(u) = 2\left(\frac{2}{x}\right)^\nu \frac{\partial}{\partial u} K_{\nu+u}(x)\,. \tag{3.41}$$
Finally, by using (3.40) and (3.41) the proof of this theorem is done. □

Remark 3.10. It should be mentioned here that by using Alzer's sharp inequality (see [2])

$$\Gamma(x+y) \leq \Gamma(x+1)\cdot\Gamma(y+1), \qquad \min\{x, y\} \geq 1,$$

and combining this with (3.35) we obtain that

$$|\mathfrak{J}_\nu(x)| \leq \frac{1}{2}\left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty |\delta_n|\Gamma(\nu+n)\left(\frac{2}{x}\right)^n \leq \frac{\Gamma(\nu+1)}{2}\left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty n!|\delta_n|\left(\frac{2}{x}\right)^n\,,$$

and the resulting power series converges in $\mathcal{I}_\delta$ for $\nu \geq 1$.

On the other hand it is worthwhile to note that, since $[x^\nu K_\nu(x)]' = -x^\nu K_{\nu-1}(x)$, the function $x \mapsto x^\nu K_\nu(x)$ is decreasing on $(0,\infty)$ for all $\nu \in \mathbb{R}$, and because of the asymptotic relation $x^\nu K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)$, where $\nu > 0$ and $x \to 0$, we obtain again the inequality (3.35). This inequality is actually the counterpart of the inequality (see [40, 51])
$$x^\nu e^x K_\nu(x) > 2^{\nu-1}\Gamma(\nu),$$
valid for all $\nu > 1/2$ and $x > 0$. Moreover, by using the classical Čebyšev integral inequality, it can be shown that (see [7]) the above lower bound can be improved as follows

$$x^{\nu-1}K_\nu(x) \geq 2^{\nu-1}\Gamma(\nu)K_1(x), \tag{3.42}$$

where $\nu \geq 1$ and $x > 0$. Summarizing, for all $x > 0$ and $\nu \geq 1$, we have the following chain of inequalities

$$\frac{1}{2}\left(\frac{2}{x}\right)^{\nu-1}\Gamma(\nu)e^{-x} < \left(\frac{2}{x}\right)^{\nu-1}\Gamma(\nu)K_1(x) \leq K_\nu(x) \leq \frac{1}{2}\left(\frac{2}{x}\right)^\nu \Gamma(\nu)\,.$$

Finally, observe that (see [7]) the inequality (3.42) is reversed when $0 < \nu \leq 1$, and this reversed inequality is actually better than (3.35) for $0 < \nu \leq 1$, that is, we have

$$x^\nu K_\nu(x) \leq 2^{\nu-1}\Gamma(\nu)xK_1(x) \leq 2^{\nu-1}\Gamma(\nu),$$

where in the last inequality we used (3.35) for $\nu = 1$. ■

Now, we are going to deduce a closed integral expression for the Neumann series $\mathfrak{X}_\nu(x)$, by using the Struve function $\mathbf{H}_\nu$.

Theorem 3.11. (Á. Baricz, D. Jankov and T.K. Pogány [5]) *Let* $\gamma \in C^1(\mathbb{R}_+)$ *and let* $\gamma|_{\mathbb{N}} = \{\gamma_n\}_{n\in\mathbb{N}}$. *Then for all*

$$x \in \mathcal{I}_\gamma = \begin{cases} \big(0, 2(e\,\ell)^{-1}\big)\,, & \text{if } -1/2 < \nu \le 1/2 \\ \big(2Le^{-1}, 2(e\,\ell)^{-1}\big)\,, & \text{if } 1/2 < \nu \le 3/2 \\ \big(4Le^{-1}, (e\,\ell)^{-1}\big)\,, & \text{if } \nu > 3/2 \end{cases}\,, \tag{3.43}$$

where

$$\ell := \limsup_{n\to\infty} n^{-1}|\gamma_n|^{1/n}, \qquad L := \limsup_{n\to\infty} n|\gamma_n|^{1/n}\,,$$

there holds

$$\begin{aligned}\mathfrak{X}_\nu(x) = \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\big((\Gamma(\nu+u+1/2) - \Gamma(\nu+u-1/2))\,\mathbf{H}_{\nu+u}(x) \\ +\Gamma(\nu+u-1/2)Y_{\nu+u}(x)\big)\cdot \mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)}\right) du\, ds\end{aligned} \tag{3.44}$$

for Neumann series of the second kind $\mathfrak{X}_\nu(x)$ *with coefficients* $\{\gamma_n\}_{n\in\mathbb{N}}$ *satisfying*

$$\ell > \begin{cases} e^{-1}, & \text{if } \nu \in (-1/2, 3/2] \\ (2e)^{-1}, & \text{if } \nu > 3/2 \end{cases}\,, \qquad L \in \begin{cases} \big(e^{-1}, 1\big)\,, & \text{if } \nu \in (-1/2, 3/2] \\ \big((2e)^{-1}, 1/2\big)\,, & \text{if } \nu > 3/2 \end{cases}\,. \tag{3.45}$$

Proof. First we establish the convergence region and related parameter constraints upon ν for $\mathfrak{X}_\nu(x)$. The Gubler–Weber formula [130, p. 165]

$$Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\int_0^1 \sin(zt)(1-t^2)^{\nu-1/2}dt + \int_0^\infty e^{-zt}(1+t^2)^{\nu-1/2}dt\right), \tag{3.46}$$

where $\Re(z) > 0$ and $\nu > -1/2$, enables the derivation of integral expression for the Neumann series of the second kind $\mathfrak{X}_\nu(x)$, by following the lines of derivation for $\mathfrak{J}_\nu(x)$. From (3.46), by means of the well–known inequality

$$(1+t^2)^{\nu-1/2} \le C_\nu(1+t^{2\nu-1}), \text{where } \; C_\nu = \begin{cases} 1, & \text{if } 1/2 < \nu \le 3/2 \\ 2^{\nu-3/2}, & \text{if } \nu > 3/2 \end{cases}\,,$$

we distinguish the following two cases.

Assuming $\nu \in (1/2, 3/2]$ we have

$$\begin{aligned} Y_\nu(x) &\le \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\int_0^1 (1-t^2)^{\nu-1/2}dt + \int_0^\infty e^{-xt}\left(1+t^{2\nu-1}\right)dt\right) \\ &= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{\sqrt{\pi}\,\Gamma(\nu+1/2)}{2\Gamma(\nu+1)} + x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}}\right) \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{1}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\left(\frac{x}{2}\right)^{\nu-1} + \frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu. \end{aligned}$$

Hence

$$|\mathfrak{X}_\nu(x)| \le \left(\frac{x}{2}\right)^\nu \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^n + \frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu-1} \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1/2)}\left(\frac{x}{2}\right)^n$$
$$+ \frac{1}{\pi}\left(\frac{2}{x}\right)^\nu \sum_{n=1}^\infty |\gamma_n|\Gamma(\nu+n)\left(\frac{2}{x}\right)^n .$$

The first two series converge uniformly in $\big(0, 2(e\,\ell)^{-1}\big)$, and the third one is uniformly convergent in $\big(2Le^{-1}, \infty\big)$. Consequently the interval of convergence becomes $\mathcal{I}_\gamma = \big(2Le^{-1}, 2(e\,\ell)^{-1}\big)$, and then the coefficients γ_n satisfy the condition $\ell \cdot L < 1$. This implies that the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup\limits_{n\to\infty} |\gamma_n|^{1/n} < 1$.

In the case $\nu > 3/2$ we have

$$\begin{aligned}
Y_\nu(x) &\le \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\int_0^1 (1-t^2)^{\nu-1/2}dt + 2^{\nu-3/2}\int_0^\infty e^{-xt}\left(1+t^{2\nu-1}\right)dt\right)\\
&= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{\sqrt{\pi}\,\Gamma(\nu+1/2)}{2\Gamma(\nu+1)} + 2^{\nu-3/2}\left(x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}}\right)\right)\\
&= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+1/2)} + \frac{2^{2\nu-3/2}\Gamma(\nu)}{\pi x^\nu} .
\end{aligned}$$

Therefore

$$|\mathfrak{X}_\nu(x)| \le \left(\frac{x}{2}\right)^\nu \sum_{n=1}^\infty \frac{|\gamma_n|}{\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^n + \frac{x^{\nu-1}}{\sqrt{2\pi}} \sum_{n=1}^\infty \frac{|\gamma_n| x^n}{\Gamma(\nu+n+1/2)}$$
$$+ \frac{1}{2\pi\sqrt{2}}\left(\frac{4}{x}\right)^\nu \sum_{n=1}^\infty |\gamma_n|\Gamma(\nu+n)\left(\frac{4}{x}\right)^n .$$

The first two series converge in $\big(0, 2(e\,\ell)^{-1}\big)$, $\big(0, (e\,\ell)^{-1}\big)$ respectively, while the third series converges uniformly for all $x > 4L/e$. This yields the interval of convergence $\mathcal{I}_\gamma = \big(4Le^{-1}, (e\,\ell)^{-1}\big)$. In this case the coefficients γ_n satisfy the constraint $4\ell L < 1$, and then the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup\limits_{n\to\infty} |\gamma_n|^{1/n} < 1/2$.

It remains the case $-1/2 < \nu \le 1/2$. Then, because of $(1+t^2)^{\nu-1/2} \le 1$, we conclude

$$Y_\nu(x) \le \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{1}{\Gamma(\nu+1/2)\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu-1},$$

and consequently $\mathcal{I}_\gamma = \big(0, 2(e\,\ell)^{-1}\big)$. Collecting these cases we get (3.43) and (3.45).

Now, let us focus on the integral representation for $\mathfrak{X}_\nu(x)$, where $x \in \mathcal{I}_\gamma$. By the Gubler–Weber formula (3.46) we have

$$\mathfrak{X}_\nu(x) = \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^\nu \sum_{n=1}^\infty \frac{\gamma_n}{\Gamma(\nu+n+1/2)}\left(\frac{x}{2}\right)^n$$
$$\cdot\left(\int_0^1 \sin(xt)(1-t^2)^{\nu+n-1/2}dt + \int_0^\infty e^{-xt}(1+t^2)^{\nu+n-1/2}\,dt\right). \tag{3.47}$$

The first expression in (3.47) we rewrite as

$$\begin{aligned}\Sigma_1(x) &= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu}\sum_{n=1}^{\infty}\frac{\gamma_n\left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+1/2)}\int_0^1 \sin(xt)(1-t^2)^{\nu+n-1/2}\mathrm{d}t\\ &= \sqrt{\frac{2x}{\pi}}\int_0^1 \sin(xt)\left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2}\mathcal{D}_\gamma(t)\,\mathrm{d}t,\end{aligned}$$

where

$$\mathcal{D}_\gamma(t) := \sum_{n=1}^{\infty}\frac{\gamma_n}{\Gamma(n+\nu+1/2)}\exp\left(-n\ln\frac{2}{x(1-t^2)}\right)$$

is the Dirichlet series analogous to one in (3.30). It is easy to see that in view of (3.45) for all $x \in \mathcal{I}_\gamma$ and $t \in (0,1)$ we have

$$\ln\frac{2}{x(1-t^2)} > 0.$$

More precisely, if $-1/2 < \nu \le 3/2$, then $x < 2(e\,\ell)^{-1}$, and

$$\frac{2}{x(1-t^2)} > \frac{e\,\ell}{1-t^2} > e\,\ell > 1\,.$$

Similarly, if $\nu > 3/2$, then $x < (e\,\ell)^{-1}$, and

$$\frac{2}{x(1-t^2)} > \frac{2e\,\ell}{1-t^2} > 2e\,\ell > 1\,.$$

Thus, the Dirichlet series' parameter is necessarily positive, and therefore $\mathcal{D}_\gamma(t)$ converges for all $x \in \mathcal{I}_\gamma$. Consequently, following the same lines as in the proof of Theorem 3.8 we deduce that

$$\Sigma_1(x) = -\int_0^\infty\int_0^{[u]}\partial_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)}\right)\frac{\partial}{\partial u}\left(\Gamma(\nu+u+1/2)\mathbf{H}_{\nu+u}(x)\right)\mathrm{d}u\,\mathrm{d}s, \tag{3.48}$$

where $\mathbf{H}_\nu$ stands for the familiar Struve function.

Below, we will simplify the second expression in (3.47):

$$\begin{aligned}\Sigma_2(x) &= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu}\sum_{n=1}^{\infty}\frac{\gamma_n\left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+1/2)}\int_0^\infty e^{-xt}(1+t^2)^{\nu+n-1/2}\,\mathrm{d}t\\ &= \sqrt{\frac{2x}{\pi}}\int_0^\infty e^{-xt}\left(\frac{x(1+t^2)}{2}\right)^{\nu-1/2}\widetilde{\mathcal{D}}_\gamma(t)\,\mathrm{d}t,\end{aligned}$$

where $\widetilde{\mathcal{D}}_\gamma(t) = \mathcal{D}_\gamma(it)$. Thus,

$$\begin{aligned}\Sigma_2(x) &= -\sqrt{\frac{2x}{\pi}}\int_0^\infty\int_0^{[u]}\mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)}\right)\\ &\quad\cdot\left(\int_0^\infty e^{-xt}\left(\frac{x(1+t^2)}{2}\right)^{\nu+u-1/2}\ln\frac{x(1+t^2)}{2}\mathrm{d}t\right)\mathrm{d}u\,\mathrm{d}s\\ &= -\pi\int_0^\infty\int_0^{[u]}\mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)}\right)\cdot\frac{\partial}{\partial u}\frac{1}{\Gamma(1/2-\nu-u)}\\ &\quad\cdot\left(\frac{2J_{-\nu-u}(x)}{\sin 2\pi(\nu+u)}-\frac{J_{\nu+u}(x)}{\sin\pi(\nu+u)}+\frac{\mathbf{H}_{\nu+u}(x)}{\cos\pi(\nu+u)}\right)\mathrm{d}u\,\mathrm{d}s\\ &= \int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\Gamma(\nu+u-1/2)\left(Y_{\nu+u}(x)-\mathbf{H}_{\nu+u}(x)\right)\mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+1/2)}\right)\mathrm{d}u\,\mathrm{d}s\,. \qquad (3.49)\end{aligned}$$

Here we applied the Euler's reflection formula $\Gamma(z)\Gamma(1-z) = \pi\mathrm{cosec}(\pi z)$, and the well–known property of the Bessel functions which was noted in equation (3.9). Summing (3.48) and (3.49) we have the desired integral representation (3.44). □

Remark 3.12. Another two linearly independent solutions of the Bessel homogeneous differential equation are *the Hankel functions* $H_\nu^{(1)}$ and $H_\nu^{(2)}$ which can be expressed as [130, p. 73]

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x) - e^{-\nu\pi i}J_\nu(x)}{i\sin(\nu\pi)}\,, \qquad (3.50)$$

$$H_\nu^{(2)}(x) = \frac{J_{-\nu}(x) - e^{\nu\pi i}J_\nu(x)}{-i\sin(\nu\pi)}\,, \qquad (3.51)$$

which build the third kind Neumann series:

$$\mathfrak{G}_\nu^{(1)}(z) := \sum_{n=1}^\infty \epsilon_n\, H_{\nu+n}^{(1)}(z), \qquad \mathfrak{G}_\nu^{(2)}(z) := \sum_{n=1}^\infty \varphi_n\, H_{\nu+n}^{(2)}(z)\,.$$

Using formulae (3.50), (3.51) we see that integral expressions for third kind Neumann series are linear combinations of similar fashion integrals achieved for $\mathfrak{N}_\nu(x)$ in Theorem A. ■

Chapter 4

Kapteyn series

The series of the type

$$\mathfrak{K}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n \, J_{\nu+n}\left((\nu+n)z\right), \qquad z \in \mathbb{C}, \tag{4.1}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν, is called a *Kapteyn series of the first kind.* Willem Kapteyn was the first who investigated such series in 1893, in his important memoir [47]. Kapteyn series have been considered in a number of mathematical physics problems. For example, the solution of the famous *Kepler's equation* [26, 62, 88]

$$E - \epsilon \sin E = M,$$

where $M \in (0, \pi)$, $\epsilon \in (0, 1]$, can be expressed *via* a Kapteyn series of the first kind:

$$E = M + 2\sum_{n=1}^{\infty} \frac{\sin(nM)}{n} J_n(n\epsilon);$$

see the integral expression for E obtained in [26, p. 1333]. That *Kepler's problem* was for the first time analytically solved by Lagrange [52], and then the solution rediscovered half a century later by Bessel in [11], in which he introduced the famous functions named after him. See also [18] for more details.

There are also *Kapteyn series of the second kind*, studied in detail e.g. by Nielsen [80]. Such series are defined by the terms consisting of a product of two Bessel function of the first kind:

$$\sum_{n=1}^{\infty} \beta_n J_{\mu+n}\left(\left(\frac{\mu+\nu}{2}+n\right)z\right) J_{\nu+n}\left(\left(\frac{\mu+\nu}{2}+n\right)z\right), \qquad z \in \mathbb{C}.$$

Summations for second kind Kapteyn series are obtained in [55, 56, 57]. More about Kapteyn series of the first and second kind one can find in [123]. Also, in [86, 87] we can find some asymptotic formulae and estimates for sums of special kind of Kapteyn series.

The importance of Kapteyn series extends from pulsar physics [55] through radiation from rings of discrete charges [56, 125], electromagnetic radiation [110], quantum modulated systems [16, 57], traffic queueing problems [20, 21] and to plasma physics problems in ambient magnetic fields [54, 111]. For more details see also the paper [124].

In [46] Kapteyn concluded, that it is possible to expand an arbitrary analytic function into a series of Bessel functions of the first kind (4.1), see for example [22, 46, 130]. Namely, let f be a function which is analytic throughout the region

$$D_a = \left\{ z \in \mathbb{C} \colon \Omega(z) = \left| \frac{z \exp\{\sqrt{1-z^2}\}}{1+\sqrt{1-z^2}} \right| \leq a \right\},$$

with $a \leq 1$. Then,

$$f(z) = \alpha_0 + 2\sum_{n=1}^{\infty} \alpha_n J_n(nz), \qquad z \in D_a,$$

where

$$\alpha_n = \frac{1}{2\pi i}\oint \Theta_n(z) f(z) dz$$

and the path of integration is the curve on which $\Omega(z) = a$. Here the function Θ_n is the so-called Kapteyn polynomial defined by

$$\Theta_0(z) = \frac{1}{z}, \quad \Theta_n(z) = \frac{1}{4}\sum_{k=0}^{[\frac{n}{2}]} \frac{(n-2k)^2 (n-k-1)!}{k!}\left(\frac{nz}{2}\right)^{2k-n}, \qquad n \in \mathbb{N}.$$

Series $\mathfrak{K}_\nu(z)$ is convergent and represents an analytic function (see [130, p. 559]) throughout the domain

$$\Omega(z) < \liminf_{n\to\infty} |\alpha_n|^{-1/(\nu+n)}.$$

But, when $z = x \in \mathbb{R}$, then the convergence region depends on the nature of the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$. This question will be tested by using Landau's bounds (2.22), (2.23) for J_ν in the proof of Theorem 4.1 below.

Motivated by the above applications in mathematical physics the main objective of this chapter is to establish two different type integral representation formulae for the Kapteyn series of the first kind. The first one is a double definite integral representation, while the second type is an indefinite integral representation formula. Also, we establish an integral representation for the special kind of Kapteyn–type series.

Let us mention, that our main findings are associated with the published paper Baricz *et al* [8] and with the unpublished paper [41].

4.1 Integral representation of Kapteyn series

In this section our aim is to deduce the double definite integral representation of the Kapteyn series $\mathfrak{K}_\nu(z)$. We shall replace $z \in \mathbb{C}$ with $x > 0$ and assume that the behavior of $(\alpha_n)_{n\in\mathbb{N}}$ ensures the convergence of the series (4.1) over a proper subset of $\mathbb{R}_+$.

Theorem 4.1. (Á. Baricz, D. Jankov and T. K. Pogány [8]) *Let* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that series* $\sum\limits_{n=1}^{\infty} n^{-1/3}\alpha_n$ *absolutely converges. Then, for all* $\nu > -3/2$ *and*

$$x \in \left(0, 2\min\left\{1, e^{-1}\big(\limsup_{n\to\infty}|\alpha_n|^{1/n}\big)^{-1}\right\}\right) =: \mathcal{I}_\alpha$$

we have the integral representation

$$\mathfrak{K}_\nu(x) = -\int_{1-\nu}^{\infty}\int_{\nu}^{[u-\nu]+\nu} \frac{\partial}{\partial u}\Big(u^{-u}\,\Gamma(u+1/2)\,J_u(ux)\Big)\,\partial_s\left(\frac{s^s\,\alpha(s-\nu)}{\Gamma(s+1/2)}\right) du\,ds\,. \tag{4.2}$$

Proof. Let us first establish the convergence conditions for the Kapteyn series of the first kind $\mathfrak{K}_\nu(x)$. For this purpose we use Landau's bounds (2.22), (2.23) for the first kind Bessel function introduced in Chapter 2. It is easy to see that there holds the estimation

$$|\mathfrak{K}_\nu(x)| \le \max\left\{b_L, \frac{c_L}{x^{1/3}}\right\}\sum_{n=1}^{\infty}\frac{|\alpha_n|}{(n+\nu)^{1/3}}\,,$$

and thus the series (4.1) converges for all $x > 0$ when $\sum\limits_{n=1}^{\infty} n^{-1/3}\alpha_n$ absolutely converges.

Now, recall the following integral representation for the Bessel function [35, p. 902]

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt\,, \qquad z\in\mathbb{C},\ \Re(\nu) > -1/2, \tag{4.3}$$

and thus, having in mind the definition of $\mathfrak{K}_\nu(x)$ in what follows we suppose that $\nu > -3/2$. Replacing (4.3) into (4.1) we have

$$\mathfrak{K}_\nu(x) = \sqrt{\frac{x}{2\pi}}\int_{-1}^{1} e^{i\nu x t}\left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2}\mathcal{D}_\alpha(t)\,dt, \qquad x > 0, \tag{4.4}$$

where $\mathcal{D}_\alpha(t)$ is the Dirichlet series

$$\mathcal{D}_\alpha(t) := \sum_{n=1}^{\infty}\frac{\alpha_n(\nu+n)^{\nu+n}}{\Gamma(n+\nu+1/2)}\,\exp\left(-n\ln\frac{2}{e^{ixt}x(1-t^2)}\right). \tag{4.5}$$

For the convergence of (4.5) we find that the related radius of convergence equals

$$\rho_{\mathfrak{K}}^{-1} = e\limsup_{n\to\infty}|\alpha_n|^{1/n}\,.$$

So, the convergence domain of $\mathcal{D}_\alpha(t)$ is $x \in (0, 2\rho_{\mathfrak{K}})$. Moreover, the Dirichlet series' parameter needs to have positive real part [48, 95], i.e.

$$\Re\left(\ln\frac{2}{e^{itx}x(1-t^2)}\right) = \ln\frac{2}{x(1-t^2)} > \ln\frac{2}{x} > 0, \qquad |t| < 1\,,$$

and hence the additional convergence range is $x \in (0, 2)$. Collecting all these estimates, we deduce that the desired integral expression exists for $x \in \mathcal{I}_\alpha$.

Expressing (4.5) first by virtue of (2.11) as the Laplace–integral, then transforming it by condensed Euler–Maclaurin formula (2.12), we get

$$\begin{aligned}\mathcal{D}_\alpha(t) &= \ln \frac{2}{e^{ixt}x(1-t^2)} \int_0^\infty \left(\frac{e^{ixt}x(1-t^2)}{2}\right)^u \sum_{n=1}^{[u]} \frac{\alpha_n(\nu+n)^{\nu+n}}{\Gamma(\nu+n+1/2)}\, du \\ &= -\int_0^\infty \int_0^{[u]} \left(\frac{e^{ixt}x(1-t^2)}{2}\right)^u \ln \frac{e^{ixt}x(1-t^2)}{2}\, \mathfrak{d}_s \left(\frac{\alpha(s)(\nu+s)^{\nu+s}}{\Gamma(\nu+s+1/2)}\right) du\, ds\,. \end{aligned} \tag{4.6}$$

Combination of (4.4) and (4.6) yields

$$\begin{aligned}\mathfrak{K}_\nu(x) &= -\sqrt{\frac{x}{2\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\alpha(s)(\nu+s)^{\nu+s}}{\Gamma(\nu+s+1/2)}\right) \\ &\quad\times \left(\int_{-1}^1 e^{ix(\nu+u)t} \left(\frac{x(1-t^2)}{2}\right)^{\nu+u-1/2} \ln \frac{e^{ixt}x(1-t^2)}{2}\, dt\right) du\, ds\,. \end{aligned} \tag{4.7}$$

Denoting

$$\mathcal{J}_x(u) := \int_{-1}^1 e^{i(\nu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+u-1/2} \ln \frac{e^{ixt}x(1-t^2)}{2}\, dt\,,$$

we have

$$\int \mathcal{J}_x(u)\, du = \sqrt{\frac{2\pi}{x}} \frac{\Gamma(\nu+u+1/2)}{(\nu+u)^{\nu+u}} J_{\nu+u}\left((\nu+u)x\right)\,,$$

that is

$$\mathcal{J}_x(u) = \sqrt{\frac{2\pi}{x}} \frac{\partial}{\partial u} \left(\frac{\Gamma(\nu+u+1/2)}{(\nu+u)^{\nu+u}} J_{\nu+u}\left((\nu+u)x\right)\right)\,. \tag{4.8}$$

Now, by virtue of (4.7) and (4.8) we conclude that

$$\mathfrak{K}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} \left(\frac{\Gamma(\nu+u+1/2)}{(\nu+u)^{\nu+u}} J_{\nu+u}\big((\nu+u)\,x\big)\right) \mathfrak{d}_s \left(\frac{\alpha(s)(\nu+s)^{(\nu+s)}}{\Gamma(\nu+s+1/2)}\right) du\, ds\,,$$

and the change of variables $\nu+t \mapsto t$, $t \in \{u,s\}$ completes the proof of (4.2). □

4.2 Another integral representation of Kapteyn series through Bessel differential equation

In the following, we deduce another integral representation for Kapteyn series (4.1), by using the fact that the Bessel functions of the first kind are solutions of the Bessel differential equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0. \tag{4.9}$$

This in turn implies that $J_{\nu+n}$ satisfies

$$x^2J''_{n+\nu}(x) + xJ'_{n+\nu}(x) + (x^2-(n+\nu)^2)J_{n+\nu}(x) = 0,$$

and thus taking $x \mapsto (\nu+n)x$ we obtain

$$x^2(\nu+n)^2 J''_{\nu+n}\,((\nu+n)x) + x(\nu+n)J'_{\nu+n}\,((\nu+n)x) + (\nu+n)^2(x^2-1)J_{\nu+n}\,((\nu+n)x) = 0\,. \quad (4.10)$$

Multiplying (4.10) by α_n, then summing up that expression in $n \in \mathbb{N}$ we arrive at

$$\begin{aligned} & x^2\mathfrak{K}''_\nu(x) + x\mathfrak{K}'_\nu(x) + (x^2-\nu^2)\mathfrak{K}_\nu(x) \\ & \qquad = \sum_{n=1}^{\infty} \big(x^2-\nu^2+(1-x^2)(\nu+n)^2\big)\alpha_n J_{n+\nu}\big((\nu+n)x\big) =: \mathfrak{L}_\nu(x); \end{aligned} \quad (4.11)$$

the right-hand side expression $\mathfrak{L}_\nu(x)$ defines the so–called *Kapteyn series of Bessel functions associated with* $\mathfrak{K}_\nu(x)$.

Our main results of this section read as follows.

Theorem 4.2. (Á. Baricz, D. Jankov and T. K. Pogány [8]) *For all* $\nu > -3/2$ *the particular solution of the nonhomogeneous Bessel–type differential equation*

$$x^2\mathfrak{z}'' + x\mathfrak{z}' + (x^2-\nu^2)\mathfrak{z} = \mathfrak{L}_\nu(x)\,, \quad (4.12)$$

with nonhomogeneous part (4.11)*, represents a Kapteyn series* $\mathfrak{z} = \mathfrak{K}_\nu(x)$ *of order* ν. *Moreover, let* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that series* $\sum\limits_{n=1}^{\infty} n^{5/3}\alpha_n$ *absolutely converges. Then, for all* $x \in \mathcal{I}_\alpha$ *we have the integral representation*

$$\begin{aligned} \mathfrak{L}_\nu(x) = &-\int_{1-\nu}^{\infty}\int_{\nu}^{[u-\nu]+\nu} \frac{\partial}{\partial u}\big(u^{-u}\,\Gamma(u+1/2)\,J_u(ux)\big) \\ &\qquad \times \mathfrak{d}_s\left(\frac{s^s\,\big((1-x^2)s^2+x^2-\nu^2\big)\;\alpha(s-\nu)}{\Gamma(s+1/2)}\right) du\,ds\,. \end{aligned} \quad (4.13)$$

Proof. Equation (4.12) we established already in the beginning of this section. Further, since associated Kapteyn series $\mathfrak{L}_\nu(x)$ is a linear combination of two Kapteyn–series, reads as follows

$$\mathfrak{L}_\nu(x) = (x^2-\nu^2)\mathfrak{K}_\nu(x) + (1-x^2)\sum_{n=1}^{\infty}(\nu+n)^2\alpha_n J_{\nu+n}\big((\nu+n)x\big)\,,$$

the uniform convergence of the second series can be easily recognized (by Landau's bounds) to be such that $\sum\limits_{n=1}^{\infty} n^{5/3}|\alpha_n| < \infty$. Making use of Theorem 4.1 with $\alpha_n \mapsto \big((1-x^2)(\nu+n)^2+x^2-\nu^2\big)\alpha_n$, we get the statement, the x–range for the integral expression (4.13) remains unchanged. □

Below, we shall need the *Bessel functions of the second kind of order* ν (or MacDonald functions) Y_ν which is defined by equation (3.9), in Chapter 2:

$$Y_\nu(x) = \operatorname{cosec}(\pi\nu)\,(J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x))\,, \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi.$$

Remember that linear combination of J_ν and Y_ν gives the particular solutions of homogeneous Bessel differential equation (4.9), when $\nu \in \mathbb{Z}$. On the other hand, when $\nu \notin \mathbb{Z}$, the particular solution is given as the linear combination of the Bessel functions of the first kind, J_ν and $J_{-\nu}$.

Theorem 4.3. (Á. Baricz, D. Jankov and T. K. Pogány [8]) *Let the situation be the same as in* Theorem 4.2. *Then we have*

$$\mathfrak{K}_\nu(x) = \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{L}_\nu(x)}{x}\,dx\right)dx + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{L}_\nu(x)}{x}\,dx\right)dx\,, \tag{4.14}$$

where $\mathfrak{L}_\nu$ is the Kapteyn series associated with the initial Kapteyn series of Bessel functions.

Proof. It is a well–known fact that J_ν and Y_ν are independent solutions of the homogeneous Bessel differential equation. Thus, the solution of the homogeneous ordinary differential equation is

$$y_h(x) = C_1Y_\nu(x) + C_2J_\nu(x)\,.$$

Since J_ν is a solution of (4.9), a guess of the particular solution is $\mathfrak{K}_\nu(x) = J_\nu(x)w(x)$. Substituting this form into homogeneous Bessel differential equation, we get

$$x^2(J_\nu''w + 2J_\nu'w' + J_\nu w'') + x(J_\nu'w + J_\nu w') + (x^2-\nu^2)J_\nu w = \mathfrak{L}_\nu(x).$$

Rewriting the equation as

$$w\big(x^2J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu\big) + w'(2x^2J_\nu' + xJ_\nu) + w''(x^2J_\nu) = \mathfrak{L}_\nu(x),$$

and using again the fact that J_ν is a solution of the homogeneous Bessel differential equation, this leads to the solution

$$w = \int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{L}_\nu\,J_\nu}{x}\,dx\right)dx + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}\,dx = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}.$$

Therefore, the desired particular solution is

$$\mathfrak{K}_\nu(x) = J_\nu(x)w(x) = J_\nu(x)\int dx\,\frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{L}_\nu\,J_\nu}{x}\,dx\right) + C_3\frac{\pi}{2}Y_\nu(x) + C_4J_\nu(x)\,.$$

Finally, as J_ν and Y_ν are independent functions that build up the solution y_h, they do not contribute to the particular solution y_p and the constants C_3, C_4 can be taken to be zero.

On the other hand, taking particular solution in the form $\mathfrak{K}_\nu(x) = Y_\nu(x)w(x)$, repeating the procedure, we arrive at

$$\mathfrak{K}_\nu(x) = Y_\nu(x)w(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{L}_\nu\,Y_\nu}{x}\,dx\right)dx - C_5\frac{\pi}{2}J_\nu(x) + C_6Y_\nu(x)\,,$$

bearing in mind that

$$\int \frac{1}{xY_\nu^2}\,dx = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}.$$

Choosing $C_5 = C_6 = 0$, we conclude the integral representation (4.14). □

4.3 Integral representation of the special kind of Kapteyn series

At the end of this chapter on Kapteyn series, we shall derive an integral representation for the special kind of Kapteyn series, i.e. for the Kapteyn–type series

$$\widetilde{K}^{\mu}_{\nu,\beta}(z) := \sum_{n=1}^{\infty} \alpha_n J_{\nu+\beta n}\big((\mu+n)z\big)\,, z \in \mathbb{C} \tag{4.15}$$

where ν, α_n are constants, $\mu \in \mathbb{C}$ and $\beta > 0$.

That representation will be useful to us in the next chapter, devoted to similar questions concerning Schlömilch series.

Theorem 4.4. (D. Jankov and T. K. Pogány [41]) *Let* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that series* $\sum\limits_{n=1}^{\infty} n^{-1/3}\alpha_n$ *absolutely converges. Then, for all* $\beta > 0$, $2(\nu+\beta)+1 > 0$ *and*

$$x \in \left(0, 2\min\left\{1, 2^{\beta-1}\,\beta^{\beta}\, e^{-\beta}\,\big(\limsup_{n\to\infty} |\alpha_n|^{1/n}\big)^{-1}\right\}\right) =: \mathcal{I}_{\alpha,\beta}$$

we have the integral representation

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = -\int_1^{\infty}\int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right) \partial_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) du\, ds. \tag{4.16}$$

Proof. Let us first establish the convergence conditions for the Kapteyn series $\widetilde{K}^{\mu}_{\nu,\beta}(x)$. For this purpose we use Landau's bounds (2.22), (2.23) for the first kind Bessel function J_ν, described in Chapter 2. It holds

$$\left|\widetilde{K}^{\mu}_{\nu,\beta}(x)\right| \le \sum_{n=1}^{\infty} |\alpha_n| \max\left\{\frac{b_L}{(\nu+\beta n)^{1/3}}, \frac{c_L}{((\mu+n)x)^{1/3}}\right\},$$

and thus the series (4.15) converges for all $x > 0$ when $\sum\limits_{n=1}^{\infty} n^{-1/3}\alpha_n$ absolutely converges.

In the following we need the integral representation of the Bessel function [35, p. 902]

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-1/2}\,dt\,, \qquad z \in \mathbb{C},\ \Re(\nu) > -1/2, \tag{4.17}$$

and thus, having in mind the definition of $\widetilde{K}^{\mu}_{\nu,\beta}(x)$ it has to be $2(\nu+\beta)+1 > 0$. Substituting (4.17) into (4.15) we get

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = \sqrt{\frac{x}{2\pi}}\int_{-1}^{1} e^{i\mu x t}\left(\frac{x(1-t^2)}{2}\right)^{\nu-1/2} \mathcal{D}_\alpha(t)\,dt, \qquad x > 0, \tag{4.18}$$

where $\mathcal{D}_\alpha(t)$ is the Dirichlet series

$$\mathcal{D}_\alpha(t) := \sum_{n=1}^{\infty} \frac{\alpha_n(\mu+n)^{\nu+\beta n}}{\Gamma(\nu+\beta n+1/2)}\, \exp\left(-n\ln\left(\frac{2}{e^{ixt/\beta}x(1-t^2)}\right)^{\beta}\right). \tag{4.19}$$

For the convergence of (4.19) we find that the related radius of convergence equals

$$\rho = \left(\frac{2\beta}{e}\right)^{\beta} \left(\limsup_{n\to\infty} |\alpha_n|^{1/n}\right)^{-1} .$$

So, the convergence domain of $\mathcal{D}_\alpha(t)$ is $|x| < \rho$. Moreover, the Dirichlet series' parameter needs to have positive real part [48, 95], i.e.

$$\Re\left(\ln \frac{2^\beta}{e^{itx} x^\beta (1-t^2)^\beta}\right) = \beta \ln \frac{2}{x(1-t^2)} > \beta \ln \frac{2}{x} > 0, \qquad |t| < 1\,,$$

and hence the additional convergence range is $x \in (0, 2)$. Collecting all these estimates, we deduce that the desired integral expression exists for $x \in \mathcal{I}_{\alpha,\beta}$.

Expressing (4.19) first as the Laplace–integral, then transforming it by condensed Euler–Maclaurin formula, we get

$$\begin{aligned}
\mathcal{D}_\alpha(t) &= \ln \frac{2^\beta}{e^{ixt}\,(x(1-t^2))^\beta} \int_0^\infty \left(e^{ixt}\left(\frac{x(1-t^2)}{2}\right)^{\beta}\right)^{u} \sum_{n=1}^{[u]} \frac{\alpha_n(\mu+n)^{\nu+\beta n}}{\Gamma(\nu+\beta n+1/2)}\, du \\
&= -\int_0^\infty \int_0^{[u]} \left(e^{ixt}\left(\frac{x(1-t^2)}{2}\right)^{\beta}\right)^{u} \ln \frac{e^{ixt}\,(x(1-t^2))^\beta}{2^\beta}\, \partial_s \left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) du\, ds\,. \qquad (4.20)
\end{aligned}$$

Combination of (4.18) and (4.20) yields

$$\begin{aligned}
\widetilde{K}^{\mu}_{\nu,\beta}(x) &= -\sqrt{\frac{x}{2\pi}} \int_0^\infty \int_0^{[u]} \partial_s \left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \\
&\quad \times \left(\int_{-1}^{1} e^{ix(\mu+u)t} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2} \ln \frac{e^{ixt}\,(x(1-t^2))^\beta}{2^\beta}\, dt\right) du\, ds. \qquad (4.21)
\end{aligned}$$

In the following, we will simplify the t–integral

$$\mathcal{J}_x(u) := \int_{-1}^{1} e^{i(\mu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2} \ln \frac{e^{ixt}\,(x(1-t^2))^\beta}{2^\beta}\, dt\,.$$

We have

$$\begin{aligned}
\int \mathcal{J}_x(u)\, du &= \int_{-1}^{1} e^{i(\mu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-1/2} dt \\
&= \sqrt{\frac{2\pi}{x}}\, \frac{\Gamma(\nu+\beta u+1/2)}{(\mu+u)^{\nu+\beta u}}\, J_{\beta u+\nu}\left((\mu+u)x\right)\,,
\end{aligned}$$

that is

$$\mathcal{J}_x(u) = \sqrt{\frac{2\pi}{x}}\, \frac{\partial}{\partial u} \left(\frac{\Gamma(\nu+\beta u+1/2)}{(\mu+u)^{\nu+\beta u}}\, J_{\beta u+\nu}\left((\mu+u)x\right)\right)\,. \qquad (4.22)$$

Now, by virtue of (4.21) and (4.22) we immediately get the integral representation (4.16). □

Chapter 5

Schlömilch series

Oscar Xavier Schlömilch introduced in 1857 in his article [108] the series of the form

$$\mathfrak{S}_\nu(z) := \sum_{n=1}^{\infty} \alpha_n \, J_\nu\left((\nu+n)z\right), \qquad z \in \mathbb{C}, \tag{5.1}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν. So, this kind series are known as *Schlömilch series* (of the order ν [1]). Rayleigh [99] has showed that such series play important roles in physics, because for $\nu = 0$ they are useful in investigation of a periodic transverse vibrations uniformly distributed in direction through the two dimensions of the membrane. Also, Schlömilch series present various features of purely mathematical interest and it is remarkable that a null–function can be represented by such series in which the coefficients are not all zero [130, p. 634].

It is worth of mention, that Schlömilch [108] proved that there exists a series $\mathfrak{S}_0^f(x)$ associated with any analytic function f. Namely, according to Watson (in renewed formulation) [130, p. 619]: *let* $f(x)$ *be an arbitrary function, with a derivative* $f'(x)$ *which is continuous in the interval* $(0, \pi)$ *and which has limited total fluctuation in this interval. Then* $f(x)$ *admits of the expansion*

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m J_0(mx) =: \mathfrak{S}_0^f(x), \tag{5.2}$$

where

$$a_0 = 2f(0) + \frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} u f'(u\sin\phi)\, d\phi\, du\,,$$
$$a_m = \frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} u f'(u\sin\phi)\cos(mu)\, d\phi\, du, \qquad m \in \mathbb{N}$$

and this expansion is valid, and the series converges in $(0, \pi)$.

We point out that this Schlömilch's result may be generalized by replacing the expansion (5.2) of order zero by $\mathfrak{S}_\nu^f(x)$ of arbitrary order ν, see [12], [79], [126] and [130, Ch. XIX.].

[1] O.X. Schlömilch considered only cases $\nu = 0, 1$.

The next generalization is suggested by the theory of Fourier series. The functions which naturally extend $\mathfrak{S}_0^f(x)$ are Bessel functions of the second kind and Struve functions. The types of series to be considered may be written in the forms:

$$\frac{\frac{1}{2}a_0}{\Gamma(\nu+1)} + \sum_{m=1}^{\infty} \frac{a_m J_\nu(mx) + b_m Y_\nu(mx)}{\left(\frac{1}{2}mx\right)^\nu},$$
$$\frac{\frac{1}{2}a_0}{\Gamma(\nu+1)} + \sum_{m=1}^{\infty} \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac{1}{2}mx\right)^\nu}.$$

Such series, with $\nu = 0$ have been considered in 1886 by Coates [17], but his proof of expanding an arbitrary functions $f(x)$ into this kind of series seems to be invalid except in some trivial case in which $f(x)$ is defined to be periodic (with period 2π) and to tend to zero as $x \to \infty$. Also for further subsequent generalizations consult e.g. Bondarenko's recent article [12] and the references therein and Miller's multidimensional expansion [68].

The series of much greater interest are direct generalization of trigonometrical series and they are called *generalized Schlömilch series*. Nielsen studied such kind of series in his memoirs consecutively in 1899 [74, 75, 76], in 1900 [77] and finally in 1901 [78, 79]. He has given the forms for the coefficients in the generalized Schlömilch expansion of arbitrary function and he has investigated the construction of Schlömlich series which represent null–functions [81, p. 348]. Filon also investigated the possibility of expanding an arbitrary function into a generalized Schlömilch series for $\nu = 0$ [29]. Using Filon's method for finding coefficients in the generalized Schlömlich expansion, Watson proved a similar fashion expansion result.

Theorem B. (G. N. Watson [130]) *Let ν be a number such that $-\frac{1}{2} < \nu < \frac{1}{2}$; and let $f(x)$ be defined arbitrarily in the interval $(-\pi, \pi)$ subject to the following conditions:* (i) *the function $h(x) = 2\nu f(x) + xf'(x) \in C^1(-\pi, \pi)$ and it has limited total fluctuation in the interval $(-\pi, \pi)$, and* (ii) *the integral*

$$\int_0^\Delta \frac{d}{dx}\left(|x|^{2\nu}\{f(x) - f(0)\}\right) dx\,, \qquad \nu \in (-1/2, 0)$$

is absolutely convergent when Δ is a (small) number either positive or negative. Then $f(x)$ admits of the expansion

$$f(x) = \frac{\frac{1}{2}a_0}{\Gamma(\nu+1)} + \sum_{m=1}^{\infty} \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac{1}{2}mx\right)^\nu},$$

where

$$\begin{aligned} a_m &= \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi}\int_0^{\frac{1}{2}\pi} \sec^{2\nu+1}\phi\, \frac{d}{d\phi}\left(\{f(u\sin\phi) - f(0)\}\sin^{2\nu}\phi\right)\cos(mu)\, d\phi\, du\,, \\ b_m &= \frac{1}{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\nu)} \int_{-\pi}^{\pi}\int_0^{\frac{1}{2}\pi} \sec^{2\nu+1}\phi\, \frac{d}{d\phi}\left(\{f(u\sin\phi) - f(0)\}\sin^{2\nu}\phi\right)\sin(mu)\, d\phi\, du\,, \end{aligned} \tag{5.3}$$

when $m > 0$; the value of a_0 is obtained by inserting an additional term $2\Gamma(\nu+1)f(0)$ on the right in the first equation of the system (5.3).

New results about Schlömilch series, which are presented in this chapter, concern to the paper by Jankov *et al.* [41].

5.1 Integral representation of Schlömilch series

In this section we will derive the double definite integral representation of the special kind of Schlömilch series

$$\mathfrak{S}_\nu^\mu(z) := \sum_{n=1}^\infty \alpha_n \, J_\nu\left((\mu+n)z\right)\,, \qquad z \in \mathbb{C}, \tag{5.4}$$

using an integral representation of Kapteyn–type series

$$\widetilde{K}_{\nu,\beta}^\mu(z) := \sum_{n=1}^\infty \alpha_n J_{\nu+\beta n}\big((\mu+n)z\big)\,, \qquad z \in \mathbb{C},\ \beta > 0\,, \tag{5.5}$$

which has been proven in the previous chapter, in Theorem 4.4.

It is easy to establish a connection between Schlömilch series (5.4) and Kapteyn–type series (5.5):

$$\mathfrak{S}_\nu^\mu(x) = \lim_{\beta\to 0} \widetilde{K}_{\nu,\beta}^\mu(x)\,. \tag{5.6}$$

Using that equality, we have the following result.

Theorem 5.1. (D. Jankov and T. K. Pogány [41]) *Let* $\alpha \in C^1(\mathbb{R}_+)$ *such that the function*

$$\kappa(u,\nu) := \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right)\partial_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right), \qquad \beta > 0$$

is integrable.
Let $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that the series* $\sum_{n=1}^\infty n^{-1/3}\alpha_n$ *absolutely converges.*
Then, for all $\nu > -1/2$ *and*

$$x \in \left(0, \min\left\{2, \big(\limsup_{n\to\infty} |\alpha_n|^{1/n}\big)^{-1}\right\}\right) =: \mathcal{I}_{\alpha,0}$$

we have the integral representation

$$\mathfrak{S}_\nu^\mu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{J_\nu\big((\mu+u)\,x\big)}{(\mu+u)^\nu}\right)\partial_s\left(\alpha(s)(\mu+s)^\nu\right) du\, ds.$$

Proof. Follows immediately from Theorem 4.4, relation (5.6) and Lebesgue's dominated convergence theorem, because using the fact that the function $\kappa(u,\nu)$ is integrable, we can establish the connection between Riemann and Lebesgue integral. □

5.2 Another integral representation of Schlömilch series

In this section our aim is to derive integral representations for Schlömlich series (5.1), using Bessel differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{5.7}$$

analogously as we did it in the previous chapter, for the Kapteyn series.

It has already been said that Bessel functions of the first kind are particular solutions of the previous equation, i.e. it holds

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0\,.$$

Now, taking $x \mapsto (\nu + n)x$ we obtain

$$x^2(\nu + n)^2 J_\nu''\left((\nu + n)x\right) + x(\nu + n)J_\nu'\left((\nu + n)x\right) + (x^2(\nu + n)^2 - \nu^2)J_\nu\left((\nu + n)x\right) = 0\,. \tag{5.8}$$

Multiplying (5.8) by α_n, and then summing up that expression in $n \in \mathbb{N}$ we get the following equality

$$x^2\mathfrak{S}_\nu''(x) + x\mathfrak{S}_\nu'(x) + (x^2 - \nu^2)\mathfrak{S}_\nu(x) \tag{5.9}$$

$$= \sum_{n=1}^{\infty} \left(1 - (\nu + n)^2\right)x^2\alpha_n J_\nu\big((\nu + n)x\big) =: \mathfrak{T}_\nu(x); \tag{5.10}$$

the right side expression $\mathfrak{T}_\nu(x)$ defines the so–called *Schlömilch series of Bessel functions associated to* $\mathfrak{S}_\nu(x)$.

Now, we can derive the following theorem:

Theorem 5.2. (D. Jankov and T. K. Pogány [41]) *Schlömilch series* (5.1) *is the solution of the nonhomogeneous Bessel–type differential equation*

$$x^2\mathfrak{y}'' + x\mathfrak{y}' + (x^2 - \nu^2)\mathfrak{y} = \mathfrak{T}_\nu(x)\,, \tag{5.11}$$

where $\mathfrak{T}_\nu(x)$ *is given with* (5.10). *Moreover, if we assume that* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and that the series* $\sum\limits_{n=1}^{\infty} n^{5/3}\alpha_n$ *absolutely converges, then for all* $x \in \mathcal{I}_{\alpha,0}$ *and* $\nu > -1/2$ *we have the integral representation*

$$\mathfrak{T}_\nu(x) = -x^2\int_1^\infty\int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{J_\nu\big((\nu + u)x\big)}{(\nu + u)^\nu}\right)\partial_s\Big(\alpha(s)\left(1 - (\nu + s)^2\right)(\nu + s)^\nu\Big)\,du\,ds\,. \tag{5.12}$$

Proof. We already showed, in the first lines of this section, that Schlömilch series (5.1) is a solution of (5.11).

Further, from (5.10) we have

$$\mathfrak{T}_\nu(x) = x^2\mathfrak{S}_\nu(x) - x^2\sum_{n=1}^{\infty}(\nu + n)^2\alpha_n\, J_\nu\left((\nu + n)x\right)\,,$$

and by Landau's bound it follows that the second series converges absolutely when $\sum\limits_{n=1}^{\infty} n^{5/3}|\alpha_n| < \infty$ absolutely converges. Using the Theorem 5.1 with $\alpha_n \mapsto \big(1-(\nu+n)^2\big)\alpha_n$, we get the integral expression (5.12). □

In what follows our aim is to present a new integral representation of the Schlömilch series (5.1), using the Bessel differential equation (5.9).

Theorem 5.3. (D. Jankov and T. K. Pogány [41]) *Let* $\alpha \in C^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = \{\alpha_n\}_{n\in\mathbb{N}}$ *and assume that series* $\sum\limits_{n=1}^{\infty} n^{5/3}\alpha_n$ *absolutely converges. Then, for all* $\nu > -1/2$ *and* $x \in \mathcal{I}_{\alpha,0}$ *we have*

$$\begin{aligned}\mathfrak{S}_\nu(x) &= \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{T}_\nu(x)}{x}\,dx\right)dx\\ &\quad + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{T}_\nu(x)}{x}\,dx\right)dx\,,\end{aligned} \tag{5.13}$$

where $\mathfrak{T}_\nu$ *is the Schlömilch series, given by* (5.10).

Proof. Let us consider the *Bessel function of the second kind of order* ν (or MacDonald function) Y_ν which is defined by

$$Y_\nu(x) = \operatorname{cosec}(\pi\nu)\left(J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)\right), \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi.$$

The homogeneous solution of the Bessel differential equation is given by

$$y_h(x) = C_1 Y_\nu(x) + C_2 J_\nu(x)\,,$$

where J_ν and Y_ν are independent solutions of the Bessel differential equation.

As J_ν is a solution, we seek for the particular solution y_p in the form $y_p(x) = J_\nu(x)w(x)$. Substituting this form into (5.9), we have

$$x^2(J_\nu'' w + 2J_\nu' w' + J_\nu w'') + x(J_\nu' w + J_\nu w') + (x^2-\nu^2)J_\nu w = \mathfrak{T}_\nu(x).$$

If we write previous equation in the following form

$$w\big(x^2J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu\big) + w'(2x^2J_\nu' + xJ_\nu) + w''(x^2J_\nu) = \mathfrak{T}_\nu(x),$$

using the fact that J_ν is a solution of the homogeneous Bessel differential equation, we get the solution

$$w = \int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{T}_\nu J_\nu}{x}\,dx\right)dx + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}dx = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}.$$

So, we have the particular solution

$$\mathfrak{S}_\nu(x) = J_\nu(x)w(x) = J_\nu(x)\int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{T}_\nu J_\nu}{x}\,dx\right)dx + C_3\frac{\pi}{2}Y_\nu(x) + C_4J_\nu(x)\,.$$

Using the fact that y_h is formed by independent functions J_ν and Y_ν, that functions do not contribute to the particular solution y_p and the constants C_3, C_4 can be taken to be zero.

Analogously, taking particular solution in the form $\mathfrak{y}_p(x) = Y_\nu(x)w(x)$ and using the equality

$$\int \frac{1}{xY_\nu^2}\,dx = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}$$

we get

$$\mathfrak{S}_\nu(x) = Y_\nu(x)w(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{T}_\nu Y_\nu}{x}\,dx\right)dx - C_5\frac{\pi}{2}J_\nu(x) + C_6Y_\nu(x)\,.$$

Again, choosing $C_5 = C_6 = 0$, we get the integral representation (5.13). □

Chapter 6

Hurwitz–Lerch Zeta function

A general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ is defined by (see, e.g. [28, p. 27, Eq. 1.11 (1)], [115, p. 121 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}\,, \tag{6.1}$$

where $a \in \mathbb{Z}\backslash\mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z|<1$; $\Re(s)>1$ when $|z|=1$. The Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ can be continued *meromorphically* to the whole complex s–plane, except for a simple pole at $s=1$ with its residue equal to 1. It is also known the integral representation [28, p. 27, Eq. 1.11 (3)]

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}\,dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(a-1)t}}{e^t - z}\,dt\,,$$

where $\Re(a)>0$; $\Re(s)>0$ when $|z|\le 1$ $(z\neq 1)$; $\Re(s)>1$ when $z=1$. Special cases of the function (6.1) one can found in the article Srivastava *et al.* [119, p. 488–489]:

- Riemann Zeta function

$$\zeta(s) := \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \Phi(1,s,1), \qquad \Re(s)>1\,;$$

- Hurwitz–Zeta function

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a), \qquad \Re(s)>1,\ a\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;$$

- Lerch Zeta function

$$l_s(\xi) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+1)^s} = \Phi(e^{2\pi i\xi},s,1), \qquad \Re(s)>1,\ \xi\in\mathbb{R}\,.$$

The general Hurwitz–Lerch Zeta function also contains some functions that are very important in *Analytic Number Theory* such as

- The Polylogarithmic (or *de Jonquère's function*)

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1),$$

defined for $s \in \mathbb{C}$, when $|z| < 1$; $\Re(s) > 1$ when $|z| = 1$ and

- the Lipschitz–Lerch Zeta function

$$\phi(\xi, s, a) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i \xi}}{(n+a)^s} = \Phi(e^{2\pi i \xi}, s, a),$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\Re(s) > 0$ when $\xi \in \mathbb{R}\backslash\mathbb{Z}$; $\Re(s) > 1$ when $\xi \in \mathbb{Z}$, which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

We note that the results presented in this chapter are mostly from the published papers Jankov *et al.* [42] (Section 6.1), Srivastava *et al.* [117] (Sections 6.2.1 and 6.2.2) and Saxena *et al.* [107] (Sections 6.3, 6.4, 6.5, 6.6 and 6.7).

6.1 Extended general Hurwitz–Lerch Zeta function

Lin and Srivastava introduced and investigated, in 2004, a generalization of the Hurwitz–Lerch Zeta function (6.1) in the following form [60, p. 727, Eq. (8)]:

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}\,. \tag{6.2}$$

Here $\mu \in \mathbb{C}$; $a, \nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\rho, \sigma \in \mathbb{R}_+$; $\rho < \sigma$ when $s, z \in \mathbb{C}$; $\rho = \sigma$ for $z \in \mathbb{C}$; $\rho = \sigma$, $s \in \mathbb{C}$ for $|z| < 1$; $\rho = \sigma$, $\Re(s - \mu + \nu) > 1$ for $|z| = 1$.

In 2008, Garg *et al.* introduced [30, Eq. (1.7)], [31] the *extended general Hurwitz–Lerch Zeta function*:

$$\Phi_{\alpha,\beta;\gamma}(z, s, r) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} \frac{z^n}{(n+r)^s}\,, \tag{6.3}$$

where $\gamma, r \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$, $s \in \mathbb{C}, \Re(s) > 0$ when $|z| < 1$ and $\Re(\gamma + s - \alpha - \beta) > 0$ when $|z| = 1$.

Our first main object is to show that the integral expression [30, Eq. (2.1)] for the extended general Hurwitz–Lerch Zeta function $\Phi_{\alpha,\beta;\gamma}(z, s, r)$ is a corollary of the integral representation formula [89, Eq. (9)] for the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series, given by Pogány.

Namely, Pogány in [89], introduced the so–called Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series

$$\mathfrak{M}_s(\mathbf{a}, \boldsymbol{\lambda}; r) = \sum_{n=0}^{\infty} \frac{a_n}{(\lambda_n + r)^s}\,, \qquad r, s > 0\,, \tag{6.4}$$

deriving closed form integral representation and bilateral bounding inequalities for $\mathfrak{M}_s(\mathbf{a}, \boldsymbol{\lambda}; r)$, generalizing at the same time some earlier results by Cerone and Lenard [15], Qi [96], Srivastava and Tomovski [121] and others.

The series (6.4) is assumed to be convergent; the sequences $\mathbf{a} := (a_n)_{n\in\mathbb{N}_0}, \boldsymbol{\lambda} := (\lambda_n)_{n\in\mathbb{N}_0}$ are positive. Following the convention that (λ_n) is monotone increasing divergent, we have

$$\boldsymbol{\lambda}: \quad 0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n \xrightarrow[n\to\infty]{} \infty\,.$$

6.1.1 Integral representation for $\Phi_{\alpha,\beta;\gamma}(z, s, r)$

Let us now derive an integral representation for the extension of general Hurwitz–Lerch Zeta function (6.3), using an integral expression for Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series.

Pogány reported [89, Theorem 1] the integral representation formula for Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series which we already introduced in Chapter 2, (2.14). Recalling the meaning of the Pochhammer symbol $(\tau)_n = \Gamma(\tau+n)/\Gamma(\tau)$, $(\tau)_0 = 1$, comparing $\mathfrak{M}_s$ (6.4) and $\Phi_{\alpha,\beta;\gamma}$ (6.3) we obtain

$$a(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{\Gamma(\alpha+x)\Gamma(\beta+x)\,z^x}{\Gamma(\gamma+x)\Gamma(x+1)}\,, \qquad \lambda(x) = \mathcal{I}(x) \equiv x,\ x \in \mathbb{R}_+, \tag{6.5}$$

where $\mathcal{I}$ denotes identical mapping. By this setting relation (2.14) becomes

$$\mathfrak{M}_s(a(x), \mathcal{I}(x); r) \equiv \Phi_{\alpha,\beta;\gamma}(z, s, r) = \frac{1}{r^s} + s\int_1^\infty\!\!\int_0^{[x]} \frac{a(u) + a'(u)\{u\}}{(r+x)^{s+1}}\,dx\,du\,, \tag{6.6}$$

where the inner u–integral is actually the Laplace integral expression of associated Dirichlet series [94]

$$\mathcal{D}_{\mathbf{a}}(x) = \sum_{n=0}^\infty a_n e^{-\lambda_n x} \equiv \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\,\frac{\left(ze^{-x}\right)^n}{n!} = {}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\,\Big|\, ze^{-x}\right].$$

Here, as usual, ${}_2F_1$ denotes the familiar Gaussian hypergeometric function. Now, it immediately follows that

$$\Phi_{\alpha,\beta;\gamma}(z, s, r) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-rx}\,{}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\,\Big|\, ze^{-x}\right]dx\,,$$

which is the same as the integral formula [30, Eq. (2.1)] by Garg *et al.* when parameter $s > 0$.

If we substitute the relation (6.5) in the integrand of (6.6) we get a new double integral expression formula for the extended general Hurwitz–Lerch Zeta function.

Theorem 6.1. (D. Jankov, T. K. Pogány and R. K. Saxena [42]) *Suppose that* $\alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{R}\backslash\mathbb{Z}_0^-, r > 0$, $\Re(s) > 0$ *and* $z \in (0, 1]$. *Then we have*

$$\begin{aligned}\Phi_{\alpha,\beta;\gamma}(z, s, r) = \frac{1}{r^s} + \frac{\Gamma(\gamma)\,s}{\Gamma(\alpha)\Gamma(\beta)}\Bigg\{&\int_1^\infty\!\!\int_0^{[x]} \frac{\Gamma(\alpha+u)\Gamma(\beta+u)\,z^u}{\Gamma(\gamma+u)\Gamma(u+1)}\,\frac{dx\,du}{(r+x)^{s+1}}\\ &+ \int_1^\infty\!\!\int_0^{[x]} \{u\}\,\frac{d}{du}\,\frac{\Gamma(\alpha+u)\Gamma(\beta+u)\,z^u}{\Gamma(\gamma+u)\Gamma(u+1)}\,\frac{dx\,du}{(r+x)^{s+1}}\Bigg\}.\end{aligned} \tag{6.7}$$

6.1.2 Bounding inequalities for $\Phi_{\alpha,\beta;\gamma}(z,s,r)$

In order to obtain sharp bilateral bounding inequalities for $\Phi_{\alpha,\beta;\gamma}(z,s,r)$, our main apparatus will be the integral representation (6.7). To employ the condition $0 \le \{u\} < 1$ in evaluating the second right–hand side integral in (6.7) we should know the monotonicity intervals of

$$b(x) := \frac{\Gamma(\alpha+x)\Gamma(\beta+x)\, z^x}{\Gamma(\gamma+x)\Gamma(x+1)}\,, \qquad x \in [1,\infty)\,.$$

Writing $\psi = \big(\ln\Gamma\big)'$, for the psi or digamma function, which was introduced in Chapter 2, we obtain

$$\begin{aligned} b\,'(x) &= b(x)\big(\psi(\alpha+x)+\psi(\beta+x)+\ln z-\psi(\gamma+x)-\psi(x+1)\big) \\ &= b(x)\sum_{k=1}^{\infty}\Big\{\frac{1}{k+x}+\frac{1}{k+\gamma+x-1} \\ &\qquad -\frac{1}{k+\alpha+x-1}-\frac{1}{k+\beta+x-1}\Big\}+b(x)\ln z\,. \end{aligned} \tag{6.8}$$

Under the assumptions $\alpha,\beta,\gamma>0$ it is not difficult to show that $b\,'(x)<0$ for all $\alpha,\beta,\gamma>0, z\in(0,1]$ when

$$\Delta := (\alpha\beta-\gamma)^2-(\alpha+\beta-\gamma-1)\big(\alpha(\beta-1)\gamma+(\alpha-\gamma)\beta\big)<0\,.$$

Thus, we arrive at the following bilateral bounding inequality result.

Theorem 6.2. (D. Jankov, T. K. Pogány and R. K. Saxena [42]) *Let α,β,γ be positive and assume that*

$$\Delta<0 \ \text{ and } \ \alpha+\beta-\gamma<\min\{1,\,\Re(s)\}\,. \tag{6.9}$$

Then for all $\Re(s), r>0$, $z\in(0,1]$ we have

$$L<\Phi_{\alpha,\beta;\gamma}(z,s,r)\le R\,, \tag{6.10}$$

where

$$R := r^{-s}+\frac{\Gamma(\gamma)s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^\infty\int_0^{[x]}\frac{\Gamma(\alpha+u)\Gamma(\beta+u)\,z^u}{\Gamma(\gamma+u)\Gamma(u+1)}\frac{dx\,du}{(r+x)^{s+1}} \tag{6.11}$$

$$L := R+\frac{\Gamma(\gamma)s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^\infty\frac{\Gamma(\alpha+[x])\Gamma(\beta+[x])\,z^{[x]}}{\Gamma(\gamma+[x])\Gamma([x]+1)}\frac{dx}{(r+x)^{s+1}}-r^{-s}\,. \tag{6.12}$$

In (6.10) *both bounds are sharp in the sense that* $0\le\{u\}<1$ *.*

Proof. Let us consider the integral (6.7). First, summing up the four fractions inside the sum in (6.8) we get in the numerator the quadratic polynomial in $X := k+x-1$:

$$P_2(X) := (\alpha+\beta-\gamma-1)X^2+2(\alpha\beta-\gamma)X+\alpha(\beta-1)\gamma+(\alpha-\gamma)\beta\,.$$

Since $\ln z\le 0$ for $z\in(0,1]$ and $\alpha+\beta-\gamma-1<0$, finally from the discriminant $\Delta<0$ we easily conclude that $b\,'(x)<0$, because $b(x)>0$ for all $x>0$, having positive α,β and γ. Recalling the convergence condition $\alpha+\beta-\gamma<\Re(s)$ of the Hurwitz–Lerch Zeta function $\Phi_{\alpha,\beta;\gamma}(z,s,r)$ we arrive at constraint (6.9) that enables suitable estimation of the integrand in (6.7).

On the other hand by the application of elementary inequality $0 \le \{u\} < 1$ to the monotone decreasing $b(u)$ we deduce $b\,'(u) < \{u\}\, b\,'(u) \le 0$ and integrating this estimate in $\big(0, [x]\big)$, we arrive at

$$b\big([x]\big) - b(0) < \int_0^{[x]} \{u\}\, b\,'(u)\, du \le 0\,.$$

Integrating the left–hand–side estimate on the range $(0, \infty)$ with respect to the measure $(r+x)^{-s-1}dx$, one gets

$$\int_1^\infty \frac{\Gamma(\alpha+[x])\Gamma(\beta+[x])\, z^{[x]}}{\Gamma(\gamma+[x])\Gamma([x]+1)} \frac{dx}{(r+x)^{s+1}} - \frac{\Gamma(\alpha)\Gamma(\beta)}{s\Gamma(\gamma) r^s} < \int_1^\infty \int_0^{[x]} \frac{\{u\}\, b\,'(u)}{(r+x)^{s+1}}\, dx\, du \le 0\,.$$

Now, employing this double–estimate to (6.7), straightforward algebra results in the assertion of Theorem 6.2. □

Now, we will need some estimates for the ψ function. Elezović *et al.* reported [27, Corollary 3] the double inequality

$$\ln\big(x+1/2\big) - \frac{1}{x} < \psi(x) < \ln\big(x+e^{-\mathbf{C}}\big) - \frac{1}{x}\,, \qquad x > 0\,,$$

such that we transform easily to

$$\ln(x+1/2) < \psi(x+1) < \ln(x+e^{-\mathbf{C}})\,, \qquad x > 0; \tag{6.13}$$

in [97, Theorem 1], it was shown that $1/2$ and $\exp\{-\mathbf{C}\}$ in (6.13) are sharp constants, see also [10].

Theorem 6.3. (D. Jankov, T. K. Pogány and R. K. Saxena [42]) *Suppose that* $\Re(s), r > 0$ *and assume that* $(\alpha, \beta, \gamma, z) \in \mathbb{R}_+^3 \times (0, 1]$ *and*

$$\big(\alpha+1/2\big)\big(\beta+1/2\big) > \big(\gamma+e^{-\mathbf{C}}\big)\big(1+e^{-\mathbf{C}}\big).$$

Then we have

$$R \le \Phi_{\alpha,\beta;\gamma}(z, s, r) < L\,. \tag{6.14}$$

Here R *and* L *are given with* (6.11) *and* (6.12).

Proof. Let $\alpha, \beta, \gamma \in \mathbb{R}_+$, $z \in (0, 1]$. Consider

$$b\,'(x) = b(x)\big(\psi(\alpha+x) + \psi(\beta+x) + \ln z - \psi(\gamma+x) - \psi(x+1)\big)\,.$$

Since $b(x) > 0$, the expression

$$\Psi(x) = \psi(\alpha+x) + \psi(\beta+x) + \ln z - \psi(\gamma+x) - \psi(x+1)$$

controls the sign of $b\,'(x)$. Making use of (6.13) we conclude

$$\Psi(x) > \ln\left(\frac{(2\alpha+2x-1)(2\beta+2x-1)z}{4(\gamma+x-1+e^{-\mathbf{C}})(x+e^{-\mathbf{C}})}\right). \tag{6.15}$$

If the right–hand–side expression in (6.15) is positive, then by obvious transformations we get

$$\begin{aligned}(2\alpha+2x-1)(2\beta+2x-1)z &> 4(\gamma+x-1+e^{-\mathbf{C}})(x+e^{-\mathbf{C}})\\ &\geq 4(\gamma+e^{-\mathbf{C}})(1+e^{-\mathbf{C}}),\end{aligned}$$

that is

$$\big(\alpha+1/2\big)\big(\beta+1/2\big) > \big(\gamma+e^{-\mathbf{C}}\big)\big(1+e^{-\mathbf{C}}\big). \tag{6.16}$$

Consequently, (6.16) suffices for $b\,'(x)>0,\ x\geq 1$. That means

$$0 \leq \int_0^{[x]} \{u\}\, b\,'(u)\, du < b\big([x]\big) - b(0).$$

Now, following the same lines of the proof of Theorem 6.2, we arrive at the asserted result (6.14). □

Finally, setting $z=1$, in the Theorem 6.2, we get another bounding inequality.

Theorem 6.4. (D. Jankov, T. K. Pogány and R. K. Saxena [42]) *Let* α,β,γ *be positive,* $\Re(s)>0$ *and assume that* $\Delta<0$ *and* $1<\alpha+\beta-\gamma<\Re(s)$*. Then for all* $r>0$ *we have*

$$R \leq \Phi_{\alpha,\beta;\gamma}(1,s,r) < L, \tag{6.17}$$

where R *and* L *are given by* (6.11) *and* (6.12).

Proof. Let us consider the integral (6.7). For $z=1$ (6.8) reduces to

$$b\,'(x) = b(x)\big(\psi(\alpha+x)+\psi(\beta+x)-\psi(\gamma+x)-\psi(x+1)\big).$$

As in the proof of Theorem 6.2, summing up the four fractions inside the sum in (6.8) we get in the numerator the same quadratic polynomial $P_2(X)$. Since $1<\alpha+\beta-\gamma<\Re(s)$ and $\Delta<0$ and $b(x)>0$ we can conclude that $b\,'(x)>0$.

The remaining part of the proof is the same as in Theorem 6.3, so we can easily get inequality (6.17). □

6.2 Extended Hurwitz–Lerch Zeta function

Srivastava *et al.*, in the article [119], investigated the family of extended Hurwitz-Lerch Zeta functions by following the ideas from an earlier paper by Lin and Srivastava [60] dealing with an interesting extension and unification of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$, its various known generalizations and investigations carried out by Gupta *et al.* [37] and others [114, 115, 116]. We recall here the definition of this already investigated family:

$$\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\, n!}\,\frac{z^n}{(n+a)^s}, \tag{6.18}$$

where $\lambda,\ \mu\in\mathbb{C}$; $a,\nu\in\mathbb{C}\backslash\mathbb{Z}_0^-$; $\rho,\sigma,\kappa\in\mathbb{R}_+$; $\kappa-\rho-\sigma>-1$ for $s,z\in\mathbb{C}$; $\kappa-\rho-\sigma=-1$ and $s\in\mathbb{C}$ for $|z|<\delta^*:=\rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa}$; $\kappa-\rho-\sigma=-1$ and $\Re(s+\nu-\lambda-\mu)>-1$ and $|z|=\delta^*$.

The extended Hurwitz–Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a)$ has the following interesting special or limit cases, which are also described in [119, p. 491–492]:

- For $\lambda = \rho = 1$ it holds
$$\Phi^{(1,\sigma,\kappa)}_{1,\mu;\nu}(z,s,a) = \Phi^{(\sigma,\kappa)}_{\mu,\nu}(z,s,a)$$
in terms of the generalized Hurwitz–Lerch Zeta function $\Phi^{(\sigma,\kappa)}_{\mu,\nu}(z,s,a)$ studied by Lin and Srivastava [60].

- If we set $\rho = \sigma = \kappa = 1$ in (6.18), we have the generalized Hurwitz–Lerch Zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a)$ studied by Garg *et al.* [30] and Jankov *et al.* [43], which is described in the previous section:
$$\Phi^{(1,1,1)}_{\lambda,\mu;\nu}(z,s,a) = \Phi^{(\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a)\,.$$

- Upon setting $\rho = \sigma = \kappa = 1$ and $\lambda = \nu$, the extended Hurwitz–Lerch Zeta function reduces to the function $\Phi^*_\mu(z,s,a)$ studied by Goyal and Laddha [34, p. 100, Eq. 1.5]:
$$\Phi^{(1,1,1)}_{\nu,\mu;\nu}(z,s,a) = \Phi^*_\mu(z,s,a)\,.$$

- For $\mu = \rho = \sigma = 1$ and $z \mapsto z/\lambda$, by the familiar principle of confluence, the limit case of (6.18), when $\lambda \to \infty$, yields the Mittag–Leffler type function $E^{(a)}_{\kappa,\nu}(s;z)$ studied by Barnes [9] (see also [28, Section 18.1]), that is
$$\lim_{\lambda\to\infty}\left\{\frac{1}{\Gamma(\nu)}\Phi^{(1,1,\kappa)}_{\lambda,1;\nu}\left(\frac{z}{\lambda},s,a\right)\right\} = \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s\cdot\Gamma(\nu+\kappa n)} =: E^{(a)}_{\kappa,\nu}(s;z)\,,$$
where $a,\nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re(\kappa) > 0$; $s,z \in \mathbb{C}$, in which the parameter $\kappa \in \mathbb{R}_+$ has been replaced, in a rather straightforward way, by $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$.

  Another two limit cases of the (6.18) are given by

- $$\Phi^{*(\sigma,\kappa)}_{\mu;\nu}(z,s,a) := \lim_{|\lambda|\to\infty}\left\{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}\left(\frac{z}{\lambda^\rho},s,a\right)\right\} = \sum_{n=0}^{\infty}\frac{(\mu)_{\sigma n}}{(\nu)_{\kappa n}\cdot n!}\frac{z^n}{(n+a)^s}\,.$$
Here $\mu \in \mathbb{C}$; $a,\nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\sigma,\kappa \in \mathbb{R}_+$; $s \in \mathbb{C}$ when $|z| < \sigma^{-\sigma}\kappa^\kappa$; $\Re(s+\nu-\mu) > 1$ when $|z| = \sigma^{-\sigma}\kappa^\kappa$; and

- $$\Phi^{*(\sigma)}_{\mu}(z,s,a) := \lim_{\min\{|\lambda|,\,|\nu|\}\to\infty}\left\{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}\left(\frac{z\nu^\kappa}{\lambda^\rho},s,a\right)\right\} = \sum_{n=0}^{\infty}\frac{(\mu)_{\sigma n}}{n!}\frac{z^n}{(n+a)^s}\,, \tag{6.19}$$
where $\mu \in \mathbb{C}$; $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $0 < \sigma < 1$ and $s,z \in \mathbb{C}$; $\sigma = 1$ and $s \in \mathbb{C}$ when $|z| < \sigma^{-\sigma}$; $\sigma = 1$ and $\Re(s-\mu) > 1$ when $|z| = \sigma^{-\sigma}$, which, for $\sigma = 1$, reduces at once to the function $\Phi^*_\mu(z,s,a)$. In fact, the function (6.19) can also be deduced as a special case of the generalized Hurwitz–Lerch function $\Phi^{(\sigma,\kappa)}_{\mu,\nu}(z,s,a)$, when $\nu = \kappa = 1$. Thus, by comparing the series definition in (6.19) with those in (6.2) and (6.18), we get the following direct relationships:
$$\Phi^{*(\sigma)}_{\mu}(z,s,a) = \Phi^{(\kappa,\sigma,\kappa)}_{\nu,\mu;\nu}(z,s,a) = \Phi^{(\sigma,1)}_{\mu,1}(z,s,a)\,.$$

Srivastava *et al.* [119] also gave a natural further generalization of the function (6.18) introducing the Fox–Wright generalized hypergeometric function ${}_p\Psi_q^*$ in the kernel, defined by (2.18), in Chapter 2. This extended Hurwitz-Lerch Zeta functions is described in the following definition.

Definition 6.5. (H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena [119]) *The family of the* extended Hurwitz–Lerch Zeta functions*:*

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a),$$

with $p+q$ *upper parameters and* $p+q+2$ *lower parameters, is given by*

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a) = \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\,\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{6.20}$$

where $p,q \in \mathbb{N}_0$*;* $\lambda_j \in \mathbb{C}$, $j=1,\cdots,p$*;* $a,\mu_j \in \mathbb{C}\setminus Z_0^-$, $j=1,\cdots,q$*;* $\rho_j,\sigma_k \in \mathbb{R}_+$, $j=1,\cdots,p$*;* $k=1,\cdots,q$*;* $\Delta > -1$ *when* $s,z\in\mathbb{C}$*;* $\Delta=-1$ *and* $s\in\mathbb{C}$ *when* $|z|<\nabla$*;* $\Delta=-1$ *and* $\Re(\Xi)>\frac{1}{2}$ *when* $|z|=\nabla$*. Here* Δ *and* ∇ *are given with* (2.19) *and* (2.20)*, respectively, and*

$$\Xi := s + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\lambda_j + \frac{p-q}{2}.$$

The special case of the function (6.20) when $p-1=q=1$ corresponds to the above–investigated generalized Hurwitz–Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a)$, defined by (6.18).

In this section, we will first derive a double-integral expression for $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$ by using a Laplace integral representation of Dirichlet series, analogously as we did in the previous section, for the extended general Hurwitz–Lerch Zeta function. Finally, by employing the so-derived integral expressions, we shall obtain extensions and generalizations of some earlier two-sided inequalities for the extended Hurwitz-Lerch Zeta function $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$.

6.2.1 First set of two-sided inequalities for $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$

In this section, we recall two theorems which will help us to derive our first set of two-sided inequalities for the extended Hurwitz-Lerch Zeta function $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$.

Our first set of main results is based essentially upon a known integral representation for $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$ due to Srivastava *et al.* [119] and the second set of results would make use of the Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-series, defined by (6.4).

First, we recall an integral expression given recently by Srivastava *et al.* [119].

Theorem C. (H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena [119]) *The following integral representation holds true:*

$$\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at}\; {}_p\Psi^*_q\left[\begin{array}{c}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q)\end{array}\Big|\, ze^{-t}\right] dt\,, \qquad \Re(a),\Re(s)>0;\ |z|<1\,, \tag{6.21}$$

provided that each member of the assertion (6.21) *exists.*

We next recall a two-sided inequality for the Fox-Wright ${}_p\Psi^*_q$-function.

Theorem D. (T. K. Pogány and H. M. Srivastava [93]) *For all* $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\rho},\boldsymbol{\sigma}) \in {}_p\mathbb{D}'_q$ *such that*

$$\lambda_j > \frac{1-\rho_j}{2}, \quad \mu_k > \frac{1-\sigma_k}{2} \quad \textit{and} \quad \lambda_j,\ \mu_k \in [0,1]\,,\ j=1,\cdots,p;\ k=1,\cdots,q\,,$$

where

$$\begin{aligned}{}_p\mathbb{D}'_q := \Bigg\{(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\rho},\boldsymbol{\sigma})\colon &\prod_{j=1}^q \left(1+\frac{\sigma_j}{\mu_j}\right)^{\sigma_j} \le \prod_{j=1}^p \left(1+\frac{\rho_j}{\lambda_j}\right)^{2\rho_j}\left(1+\frac{1}{\lambda_j}\right)^{-\rho_j^2}\\ \textit{and} \qquad &\prod_{j=1}^p \sqrt{\lambda_j}(\lambda_j+\rho_j)^{\rho_j-\frac12} \le \prod_{j=1}^q\left(\mu_j - \frac{1-\sigma_j}{2}\right)^{\sigma_j}\Bigg\},\end{aligned}$$

the following two-sided inequality holds true:

$$e^{\Omega^*\cdot|z|} \le {}_p\Psi^*_q\left[\begin{array}{c}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p)\\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q)\end{array}\Big|\, z\right] \le 1-\Omega^*\cdot\left(1-e^{|z|}\right), \tag{6.22}$$

$$\Omega^* := \frac{\prod\limits_{j=1}^p (\lambda_j)_{\rho_j}}{\prod\limits_{j=1}^q (\mu_j)_{\sigma_j}} < 1;\ z\in\mathbb{R}\,.$$

Remark 6.6. Condition $\Omega^* < 1$ is necessary for the existence of the inequality (6.22). ■

We are now ready to state and prove our first main result in this section.

Theorem 6.7. (H. M. Srivastava, D. Jankov, T. K. Pogány and R. K. Saxena [117]) *Assume that* $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\rho},\boldsymbol{\sigma}) \in {}_p\mathbb{D}'_q$, $p,q\in\mathbb{N}_0$ *and* $s,a\in\mathbb{R}_+$. *Then*

$$\frac{1}{a^s} + \frac{\Omega^*\cdot|z|}{(a+1)^s} \le \Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a) \le R\,, \tag{6.23}$$

where

$$R = \frac{1-\Omega^*}{a^s} + \Omega^*\cdot\Phi^{(0,1)}_{\lambda;1}(|z|,s,a) \le \frac{1}{a^s}\left(1-\Omega^*\cdot(1-e^{|z|})\right).$$

Proof. First of all, we prove the left-hand side of the inequality (6.23). Using (6.21) from Theorem C, (6.22) from Theorem D and the following well-known inequality:

$$e^x \ge 1+x\,,$$

we easily get

$$\begin{aligned}\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a) &\geq \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at}\, \exp\left(\Omega^*\cdot|z|e^{-t}\right)dt \geq \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at}\left(1+\Omega^*\cdot|z|e^{-t}\right)dt\\ &= \frac{1}{\Gamma(s)}\left(\int_0^\infty t^{s-1}\, e^{-at}\,dt + \Omega^*\cdot|z|\int_0^\infty t^{s-1}\, e^{-(a+1)t}\,dt\right)\\ &= \frac{1}{a^s} + \frac{\Omega^*\cdot|z|}{(a+1)^s}\,.\end{aligned}$$

Hence

$$\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a) \geq \frac{1}{a^s} + \frac{\Omega^*\cdot|z|}{(a+1)^s}\,.$$

Now, it remains to prove the right-hand upper bound (6.23). Indeed, by virtue of the upper bound in (6.22), we estimate the integrand in (6.21) and deduce that

$$\begin{aligned}\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a) &\leq \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, e^{-at}\left(1-\Omega^*\cdot\left(1-\exp\left\{|z|e^{-t}\right\}\right)\right)dt\\ &= \frac{1}{\Gamma(s)}\left(\left(1-\Omega^*\right)\int_0^\infty t^{s-1}\, e^{-at}\,dt + \Omega^*\int_0^\infty t^{s-1}e^{-at}\,\exp\left\{|z|e^{-t}\right\}dt\right)\\ &= \frac{1-\Omega^*}{a^s} + \frac{\Omega^*}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,\exp\left(|z|e^{-t}\right)dt\,. \qquad (6.24)\end{aligned}$$

Expanding $\exp\left(|z|e^{-t}\right)$ into its Taylor-Maclaurin series and applying the readily justified interchange of summation and integration, we conclude that

$$\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a) \leq \frac{1-\Omega^*}{a^s} + \Omega^*\sum_{n=0}^\infty \frac{1}{n!}\frac{|z|^n}{(a+n)^s}\,,$$

where the sum can easily be recognized as follows:

$$\sum_{n=0}^\infty \frac{1}{n!}\frac{|z|^n}{(a+n)^s} = \Phi_{\lambda;1}^{(0,1)}(|z|,s,a)\,,$$

that is, a special case of the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a)$ with $p=q=1$.

The upper bound in (6.23) contains a generalized Hurwitz-Lerch Zeta function term. However, upon estimating the integrand in (6.24) by means of the following rather elementary inequality:

$$e^{-t}\leq 1\,, \qquad t\geq 0\,,$$

we get the following remarkably simple result:

$$\Phi_{\lambda;\mu}^{(\rho,\sigma)}(z,s,a) \leq \frac{1-\Omega^*}{a^s} + \frac{\Omega^*}{a^s}\,e^{|z|}\,.$$

This completes the proof of the Theorem 6.7. □

Remark 6.8. In the proof of Theorem 6.7 there are used the elementary inequalities $e^x \geq 1+x$ and $e^{-x}\leq 1$, where $x\geq 0$. Using certain sharper inequalities (see for example the *Analytic Inequalities* [70] monograph by Mitrinović) for e^x and e^{-x}, it would be possible significantly to improve the bilateral bound in (6.23). ■

6.2.2 Second set of two-sided bounding inequalities for $\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a)$

We already mentioned the following integral representation formula, which was reported by Pogány [89, Theorem 1], but here it is listed in a slightly different labels:

$$\mathfrak{M}_s(\mathbf{b},\boldsymbol{\eta};r) = \frac{b_0}{r^s} + s\int_{\eta_1}^{\infty}\int_0^{[\eta^{-1}(x)]} \frac{b(u)+b\,'(u)\{u\}}{(r+x)^{s+1}}\,dx\,du \tag{6.25}$$

where $b\in C^1[0,\infty)$; $b(u)\big|_{u=\mathbb{N}_0} =: \mathbf{b}$, $\eta^{-1}(x)$ stands for the inverse of the function $\eta(x)$ and the series $\mathfrak{M}_s(\mathbf{b},\boldsymbol{\eta};r)$ is assumed to be convergent.

Comparing $\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a)$ and $\mathfrak{M}_s(\mathbf{b},\boldsymbol{\eta};r)$, we find for all $x\in\mathbb{R}_+$ that

$$b(x) = \left(\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j x}}{\prod_{j=1}^{q}(\mu_j)_{\sigma_j x}}\right)\frac{z^x}{\Gamma(x+1)} \quad \text{and} \quad \eta(x)=\mathcal{I}(x)\equiv x, \tag{6.26}$$

where $\mathcal{I}$ denotes the identical mapping. By this setting, the integral representation in (6.25) assumes the following form:

$$\mathfrak{M}_s\big(b(x),\mathcal{I}(x);a\big) \equiv \Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a) = \frac{1}{a^s} + s\int_1^{\infty}\int_0^{[x]} \frac{b(u)+b\,'(u)\{u\}}{(a+x)^{s+1}}\,dx\,du\,. \tag{6.27}$$

If we substitute from the relation (6.26) in the integrand of (6.27), we get a new double integral expression formula for the extended Hurwitz-Lerch Zeta function $\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a)$, given by Theorem 6.9 below.

Theorem 6.9. (H. M. Srivastava, D. Jankov, T. K. Pogány and R. K. Saxena [117]) *Assume that* $\lambda_j\in\mathbb{C}$, $a,\mu_k\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $\rho_j,\sigma_k\in\mathbb{R}_+$, $j=1,\cdots,p$; $k=1,\cdots,q$, $\min\{s,\Re(a)\}>0$ *and* $0<z\le 1$. *Then*

$$\begin{aligned}\Phi^{(\rho,\sigma)}_{\lambda;\mu}(z,s,a) = \frac{1}{a^s} + s\,\Bigg(&\int_1^{\infty}\int_0^{[x]} \frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j u}}{\prod_{j=1}^{q}(\mu_j)_{\sigma_j u}}\,\frac{z^u}{\Gamma(u+1)}\,\frac{dx\,du}{(a+x)^{s+1}}\\ &+\int_1^{\infty}\int_0^{[x]}\{u\}\,\frac{d}{du}\left\{\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j u}}{\prod_{j=1}^{q}(\mu_j)_{\sigma_j u}}\,\frac{z^u}{\Gamma(u+1)}\right\}\frac{dx\,du}{(a+x)^{s+1}}\Bigg)\,.\end{aligned} \tag{6.28}$$

In the remaining part of this chapter, our main apparatus will be the above integral representation. To apply the inequality $0\le\{u\}<1$ in evaluating the integral in (6.28), we should know the monotonicity behavior of the function $h(x)$ given by

$$h(x) = \left(\frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{\prod_{j=1}^{p}\Gamma(\lambda_j)}\right)\left(\frac{\prod_{j=1}^{p}\Gamma(\lambda_j+\rho_j x)}{\prod_{j=1}^{q}\Gamma(\mu_j+\sigma_j x)}\right)\frac{z^x}{\Gamma(x+1)}\,, \qquad x\ge 1\,. \tag{6.29}$$

In the next theorem, again, we need Psi function $\psi(z)$, and the Euler-Mascheroni constant $\mathbf{C}$.

Theorem 6.10. (H. M. Srivastava, D. Jankov, T. K. Pogány and R. K. Saxena [117]) *Consider the parameters* $a, s \in \mathbb{R}_+$ *and let* $p, q \in \mathbb{N}_0$, λ_j, μ_k, ρ_j, $\sigma_k \in \mathbb{R}$, $j = 1, \cdots, p$; $k = 1, \cdots, q$. *Then each of the following two–sided inequalities holds true:*

$$L < \Phi_{\lambda;\mu}^{(\rho,\sigma)}(z, s, a) \leq R\,, \tag{6.30}$$

for $\mu_{k_j} + x\sigma_{k_j} \geq \lambda_j + x\rho_j > 0$; $\sigma_{k_j} \geq \rho_j > 0$; $\psi(\lambda_j + x\rho_j) > 0$; $x > 0$; $z \in (0, e^{-\mathbf{C}})$; $p \leq q$, *where* $(k_1, \cdots, k_p)$ *is a permutation of* p *indices* $k_j \in \{1, \cdots, q\}$ *and*

$$R := a^{-s} + s \int_1^\infty \int_0^{[x]} \frac{\prod_{j=1}^p (\lambda_j)_{\rho_j u}}{\prod_{j=1}^q (\mu_j)_{\sigma_j u}} \frac{z^u}{\Gamma(u+1)} \frac{dx\, du}{(a+x)^{s+1}}$$

$$L := R + s \int_1^\infty \frac{\prod_{j=1}^p (\lambda_j)_{\rho_j [x]}}{\prod_{j=1}^q (\mu_j)_{\sigma_j [x]}} \frac{z^{[x]}}{\Gamma([x]+1)} \frac{dx}{(a+x)^{s+1}} - a^{-s}\,;$$

moreover

$$R \leq \Phi_{\lambda;\mu}^{(\rho,\sigma)}(z, s, a) < L\,, \tag{6.31}$$

when $\lambda_{j_k} + x\rho_{j_k} \geq \mu_k + x\sigma_k > 0$; $\rho_{j_k} \geq \sigma_k > 0$; $\psi(\mu_k + x\sigma_k) > 0$; $x > 0$; $z > x + e^{-\mathbf{C}}$; $p \geq q$.

Here $(j_1, \ldots, j_q)$ *stands for a* q*–tiple of indices from* $\{1, \ldots, p\}$. *The upper bound in* (6.30) *and the lower bound in* (6.31) *are sharp in the sense that* $0 \leq \{u\} < 1$.

Proof. To prove the assertions (6.30) and (6.31) of Theorem 6.10, we consider the function $g(x)$ given by

$$g(x) = \left(\frac{\prod_{j=1}^p \Gamma(\lambda_j + x\rho_j)}{\prod_{j=1}^q \Gamma(\mu_j + x\sigma_j)} \right) \frac{z^x}{\Gamma(x+1)}\,,$$

so that, obviously, the monotonicity of $g(x)$ implies the monotonicity of the function $h(x)$ defined by (6.29). Hence

$$g'(x) = g(x) \left(\sum_{j=1}^p \rho_j \psi(\lambda_j + x\rho_j) + \ln z - \sum_{j=1}^q \sigma_j \psi(\mu_j + x\sigma_j) - \psi(x+1) \right).$$

Since $g(x) > 0$ for all $x > 0$, the following expression:

$$f(x) = \sum_{j=1}^p \rho_j \psi(\lambda_j + x\rho_j) + \ln z - \sum_{j=1}^q \sigma_j \psi(\mu_j + x\sigma_j) - \psi(x+1)$$

controls the sign of $g'(x)$. As $p \le q$, transforming $f(x)$ into

$$f(x) = \sum_{j=1}^{p} \big(\rho_j \psi(\lambda_j + x\rho_j) - \sigma_{k_j}\psi(\mu_{k_j} + x\sigma_{k_j})\big) + \big(\ln z - \psi(1+x)\big) - \sum_{\substack{j=1 \\ j\neq k_1,\cdots,k_p}}^{q} \sigma_j \psi(\mu_j + x\sigma_j)\,,$$

and then, using the fact that $z \in (0, e^{-\mathbf{C}})$, we find that $f(x) < 0$, i.e. $g'(x) < 0, x > 0$, because the Psi function $\psi(x)$ is increasing for $x > 0$, that is,

$$\psi(x+1) > \psi(1) = -\mathbf{C}.$$

From the elementary inequalities $0 \le \{u\} < 1$ and $g'(u) < 0$, we deduce that

$$h'(u) < \{u\}h'(u) \le 0$$

and, upon integrating this estimate in $\big(0, [x]\big)$, we have

$$h\big([x]\big) - h(0) < \int_0^{[x]} \{u\}h'(u)\,du \le 0\,.$$

Integrating the left-hand side estimate on the range $\mathbb{R}_+$ with respect to the measure $(a+x)^{-s-1}\,dx$, we arrive at the assertion (6.30) of Theorem 6.10. Similarly, by the following inequality [27, Corollary 3]

$$\psi(x+1) < \ln(x + e^{-\mathbf{C}})\,, \qquad x > 0\,,$$

we see for the function $h(x)$ given by (6.29) that $h'(x) > 0, x > 0$. The remaining part of the proof of (6.31) is the same as the proof of the assertion (6.30), so we easily get the inequality (6.31). □

6.3 On generalized Hurwitz–Lerch Zeta distributions occuring in statistical inference

The object of this section is to define certain new incomplete generalized Hurwitz–Lerch Zeta functions and incomplete generalized Gamma functions. Further, we introduce two new statistical distributions named as, generalized Hurwitz–Lerch Zeta Beta prime distribution and generalized Hurwitz–Lerch Zeta Gamma distribution and investigate their statistical functions, such as moments, distribution and survivor function, characteristic function, the hazard rate function and the mean residue life functions. Finally, Moment Method parameter estimators are given by means of a statistical sample of size n. The result obtained provide an elegant extension of the work reported earlier by Garg *et al.* [31] and others.

Special attention will be given to the special case of extended Hurwitz–Lerch Zeta function, given by (6.18), which we mentioned earlier in (6.19):

$$\Phi^*_\mu(z,s,a) := \Phi^{(1,1,1)}_{1,\mu,1}(z,s,a) = \sum_{n=1}^{\infty} \frac{(\mu)_n}{(n+a)^s}\,\frac{z^n}{n!}\,.$$

Moreover, the article [119] contains the integral representation

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_2\Psi_1^*\Big[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\,ze^{-t}\Big]dt, \tag{6.32}$$

valid for all $a, s \in \mathbb{C}$, $\Re(a) > 0$, $\Re(s) > 0$, when $|z| \le 1$, $z \ne 1$; and $\Re(s) > 1$ for $z = 1$. Here ${}_p\Psi_q^*[\cdot]$ stands for the *unified variant of the celebrated Fox–Wright generalized hypergeometric function* defined by (2.18), in Chapter 2.

Finally, we recall the integral expression for function $\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$, derived by Srivastava *et al.* [119]:

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_0^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa},s,a\Big)\,dt, \tag{6.33}$$

where $\Re(\nu) > \Re(\lambda) > 0$, $\kappa \ge \rho > 0$, $\sigma > 0, s \in \mathbb{C}$.

Now, we study generalized incomplete functions and the associated statistical distributions based mainly on integral expressions (6.32) and (6.33).

6.4 Families of incomplete φ and ξ functions

By virtue of integral (6.33), we define the *lower incomplete generalized Hurwitz–Lerch Zeta function* as

$$\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_0^x \frac{t^{\lambda-1}}{(1+t)^\nu}\,\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa},s,a\Big)dt, \tag{6.34}$$

and the *upper (complementary) generalized Hurwitz–Lerch Zeta function* in the form

$$\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_x^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\,\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa},s,a\Big)dt. \tag{6.35}$$

In both cases one requires $\Re(\nu), \Re(\lambda) > 0$, $\kappa \ge \rho > 0$; $\sigma > 0$, $s \in \mathbb{C}$.

From (6.34) and (6.35) readily follows that

$$\begin{aligned}\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) &= \lim_{x\to\infty}\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \lim_{x\to 0+}\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)\,,\\ \Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) &= \varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) + \overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)\,, \qquad x \in \mathbb{R}_+.\end{aligned}$$

In view of the integral expression (6.32), the *lower incomplete generalized Gamma function* and the *upper (complementary) incomplete generalized Gamma function* are defined respectively by

$$\xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)}\int_0^x t^{s-1}e^{-at}\,{}_2\Psi_1^*\Big[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\,ze^{-bt}\Big]dt \tag{6.36}$$

and

$$\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)}\int_x^\infty t^{s-1}e^{-at}{}_2\Psi_1^*\Big[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\,ze^{-bt}\Big]dt, \tag{6.37}$$

where $\Re(a), \Re(s) > 0$, when $|z| \le 1\,(z \ne 1)$ and $\Re(s) > 1$, when $z = 1$, provided that each side exists. By virtue of (6.36) and (6.37) we easily conclude the properties:

$$\Phi^{(\rho,\sigma,\rho)}_{\lambda,\mu,\nu}(z,s,a/b) = \lim_{x\to\infty} \xi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a,b|x) = \lim_{x\to 0+} \overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a,b|x)\,,$$
$$\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b) = \xi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a,b|x) + \overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a,b|x)\,, \qquad x \in \mathbb{R}_+.$$

6.5 Generalized Hurwitz–Lerch Zeta Beta prime distribution

Special functions and integral transforms are useful in the development of the theory of probability density functions (PDF). In this connection, one can refer to the books e.g. by Mathai and Saxena [63, 64] or by Johnson and Kotz [44, 45]. Hurwitz–Lerch Zeta distributions are studied by many mathematicians such as Dash, Garg, Gupta, Kalla, Saxena, Srivastava etc. (see e.g. [30, 31, 37, 38, 103, 104, 105, 106, 120]). Due to usefulness and popularity of Hurwitz–Lerch Zeta distribution in reliability theory, statistical inference etc. we are motivated to define a generalized Hurwitz–Lerch Zeta distribution and to investigate its important properties.

Let the random variable X be defined on some fixed standard probability space $(\Omega, \mathfrak{F}, P)$. The r.v. X such that possesses PDF

$$f(x) = \begin{cases} \dfrac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^{\nu}} \, \dfrac{\Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\dfrac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}, \tag{6.38}$$

we call *generalized Hurwitz–Lerch Zeta Beta prime* and write $X \sim$ HLZB$'$. Here μ, λ are shape parameters, and z stands for the scale parameter which satisfy $\Re(\nu) > \Re(\lambda) > 0$, $s \in \mathbb{C}$, $\kappa \ge \rho > 0$, $\sigma > 0$.

The behavior of the PDF $f(x)$ at $x = 0$ depends on λ in the manner that $f(0) = 0$ for $\lambda > 1$, while $\lim_{x\to 0+} f(x) = \infty$ for all $0 < \lambda < 1$.

Now, let us mention some interesting special cases of PDF (6.38).

- For $\sigma = \rho = \kappa = 1$ we get the following Hurwitz–Lerch Zeta Beta prime distribution discussed by Garg *et al.* [31]:

$$f_1(x) = \begin{cases} \dfrac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\Phi_{\lambda,\mu,\nu}(z,s,a)} \, \dfrac{x^{\lambda-1}}{(1+x)^{\nu}} \, \Phi^*_{\mu}\Big(\dfrac{zx}{1+x}, s, a\Big), & \text{if } x > 0 \\ 0, & \text{elsewhere} \end{cases},$$

where $a \notin \mathbb{Z}_0^-$, $\Re(\nu) > \Re(\lambda) > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re(s-\mu) > 0$, when $|z| = 1$. Here $\Phi^*_{\mu}(\cdot, s, a)$ stands for the Goyal–Laddha type generalized Hurwitz–Lerch Zeta function described in (6.19).

- If we set $\sigma = \rho = \kappa = \lambda = 1$ it gives a new probability distribution function, defined by

$$f_2(x) = \begin{cases} \dfrac{\nu - 1}{(1+x)^{\nu}\, \Phi_{1,\mu,\nu}(z,s,a)}\, \Phi^*_{\mu}\Big(\dfrac{zx}{1+x}, s, a\Big), & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases},$$

where $a \notin \mathbb{Z}_0^-$, $\Re(\nu) > 1$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re(s-\mu) > 0$, when $|z| = 1$.

- When $\sigma = \rho = \kappa = 1$ and $\nu = \mu$, from (6.38) it follows

$$f_3(x) = \begin{cases} \dfrac{\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\mu-\lambda)\Phi^*_{\lambda}(z,s,a)}\, \dfrac{x^{\lambda-1}}{(1+x)^{\mu}}\, \Phi^*_{\mu}\Big(\dfrac{zx}{1+x}, s, a\Big), & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases},$$

with $a \notin \mathbb{Z}_0^-$, $\Re(\mu) > \Re(\lambda) > 0$, $x \in \mathbb{R}$, $s\ in \mathbb{C}$ when $|z| < 1$ and $\Re(s-\mu) > 0$, when $|z| = 1$.

- For $\sigma = \rho = \kappa = 1$ and $\mu = 0$, we obtain the Beta prime distribution (or the Beta distribution of the second kind.

- For Fischer's F–distribution, which is a Beta prime distribution, we set $\sigma = \rho = \kappa = 1$ and replace $x = mx/n$, $\lambda = m/2$, $\nu = (m+n)/2$, where m and n are positive integers.

6.5.1 Statistical functions for the HLZB′ distribution

In this section we introduce some classical statistical functions for the HLZB′ distributed random variable having the PDF given with (6.38). These characteristics are *moments* of positive, fractional order $m_r, r \in \mathbb{R}$, being the Mellin transform of order $r+1$ of the PDF; the *generating function* $G_X(t)$ which equals to the Laplace transform and the *characteristic function* (CHF) $\phi_X(t)$ which coincides with the Fourier transform of the PDF (6.38).

We point out that all three highly important characteristics of the probability distributions can be uniquely expressed *via* the operator of the mathematical expectation E. However, it is well–known that for any Borel function ψ there holds

$$\mathsf{E}\psi(X) = \int_{\mathbb{R}} \psi(x) f(x)\, dx. \tag{6.39}$$

To obtain explicitely $m_r, G_X(t), \phi_X(t)$ we also need in the sequel the extended Hurwitz–Lerch Zeta function, already introduced in (6.20).

Theorem 6.11. (R. K. Saxena, T. K. Pogány, R. Saxena and D. Jankov [107]) *Let* $X \sim$ HLZB′ *be a r.v. defined on a standard probability space* $(\Omega, \mathfrak{F}, \mathsf{P})$ *and let* $r \in \mathbb{R}_+$. *Then the* r*th fractional order moment of* X *reads as follows*

$$m_r = \frac{(\lambda)_r \sin \pi(\nu-\lambda)}{(1-\nu+\lambda)_r \sin \pi(\nu-\lambda-r)}\, \frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}. \tag{6.40}$$

Proof. The fractional moment m_r of the r.v. $X \sim$ HLZB$'$ is given by

$$m_r = \mathsf{E}X^r = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^\infty \frac{x^{\lambda+r-1}}{(1+x)^\nu} \Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zx^\rho}{(1+x)^\kappa}, s, a\Big)\, dx\,, \qquad r \in \mathbb{R}_+\,,$$

where A is the related normalizing constant.

Expressing the Hurwitz–Lerch Zeta function in initial power series form, and interchanging the order of summation and integration, we find that:

$$\begin{aligned} m_r &= \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}}{(\nu-\lambda)_{(\kappa-\rho)n}} \frac{z^n}{(n+a)^s\, n!} \int_0^\infty \frac{x^{\lambda+r+\rho n-1}}{(1+x)^{\nu+\kappa n}}\, dx \\ &= \frac{A\Gamma(\lambda+r)\Gamma(\nu-\lambda-r)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}}{(\nu)_{\kappa n}} \cdot \frac{(\nu-\lambda-r)_{(\kappa-\rho)n}}{(\nu-\lambda)_{(\kappa-\rho)n}} \frac{z^n}{(n+a)^s\, n!}\,. \end{aligned}$$

By the Euler's reflection formula we get

$$\begin{aligned} m_r &= \frac{A(\lambda)_r\,\Gamma(1-\nu+\lambda)\sin\pi(\nu-\lambda)}{\Gamma(1-\nu+\lambda+r)\sin\pi(\nu-\lambda-r)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}\,(n+a)^s\, n!} \\ &= \frac{A(\lambda)_r\, \sin\pi(\nu-\lambda)}{(1-\nu+\lambda)_r\, \sin\pi(\nu-\lambda-r)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}(n+a)^s\, n!}\,, \end{aligned}$$

which is same as (6.40). □

We point out that for the integer $r \in \mathbb{N}$, the moment (6.40) it reduces to

$$m_r = \frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r} \frac{\Phi_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}(z,s,a)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}\,. \tag{6.41}$$

Theorem 6.12. (R. K. Saxena, T. K. Pogány, R. Saxena and D. Jankov [107]) *The generating function* $G_X(t)$ *and the* CHF $\phi_X(t)$ *for the r.v.* $X \sim$ HLZB$'$, *for all* $t \in \mathbb{R}$, *are represented in the form*

$$\begin{aligned} G_X(t) &= \mathsf{E}\, e^{-tX} = \frac{1}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)} \sum_{r=0}^\infty \frac{(\lambda)_r}{(1+\lambda-\nu)_r} \frac{t^r}{r!} \Phi_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}^{(\sigma,\rho,\kappa-\rho;\kappa,\kappa-\rho)}(z,s,a), \\ \phi_X(t) &= \mathsf{E}\, e^{itX} = \frac{1}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)} \sum_{r=0}^\infty \frac{(\lambda)_r}{(1+\lambda-\nu)_r} \frac{(-it)^r}{r!} \Phi_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}^{(\sigma,\rho,\kappa-\rho;\kappa,\kappa-\rho)}(z,s,a)\,. \end{aligned}$$

Proof. Setting $\psi(X) = e^{-tX}$ in (6.39) respectively, then expanding the Laplace kernel into Maclaurin series, by legitimate interchange the order of summation and integration we obtain the generating function $G_X(t)$ in terms of (6.41), using also the relation (2.5). Because $\phi_X(t) = G_X(-it), t \in \mathbb{R}$, the proof is completed. □

The second set of important statistical functions concers the reliability applications of the newly introduced generalized Hurwitz–Lech Zeta Beta prime distribution. The functions associated with r.v. X are the cumulative distribution function (CDF) F, the survivor function $S = 1 - F$, the hazard rate function $h = f/(1-F)$, and the mean residual life function $K(x) = \mathsf{E}[X - x | X \geq x]$. Their explicit formulæ are given in terms of lower and upper incomplete (complementary) φ–functions.

Theorem 6.13. (R. K. Saxena, T. K. Pogány, R. Saxena and D. Jankov [107]) *Let r.v.* $X \sim \mathrm{HLZB}'$. *Then we have:*

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\,\frac{x^{\lambda-1}}{(1+x)^{\nu}}\,\frac{\Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\frac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}, \tag{6.42}$$

$$\begin{aligned} K(x) &= \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\,\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)} \sum_{n=0}^{\infty} \frac{(\mu))_{\sigma n}}{(\nu-\lambda)_{(\kappa-\rho)n}}\,\frac{z^n}{(n+a)^s\, n!} \\ &\quad \times \mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big) - x, \end{aligned} \tag{6.43}$$

where

$$\mathrm{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,\mathrm{d}t\,, \qquad \min\{\Re(a), \Re(b)\} > 0,\ |z| < 1$$

represents the incomplete Beta*–function.*

Proof. The CDF and the survivor functions of the r.v. X are

$$F(x) = \frac{\varphi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}, \qquad S(x) = \frac{\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}\,, \qquad x > 0\,,$$

and vanish elsewhere. Therefore, being $h(x) = f(x)/S(x)$, (6.42) is proved.

It is well–known that for the mean residual life function there holds [36]

$$K(x) = \frac{1}{S(x)}\int_x^{\infty} tf(t)\mathrm{d}t - x\,.$$

The integral will be

$$\mathcal{J} = \int_x^{\infty} tf(t)\mathrm{d}t = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\, n!} \int_x^{\infty} \frac{t^{\lambda+\rho n}}{(1+t)^{\nu+\kappa n}}\,\mathrm{d}t\,,$$

where the innermost t–integral reduces to the incomplete Beta function in the following way:

$$\int_x^{\infty} \frac{t^{p-1}}{(1+t)^q}\,\mathrm{d}t = \int_0^{(1+x)^{-1}} t^{q-p-1}(1-t)^{p-1}\,\mathrm{d}t = \mathrm{B}_{(1+x)^{-1}}(q-p,p)\,.$$

Therefore we conclude

$$\mathcal{J} = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\, n!}\, \mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big)\,.$$

After some simplification it leads to the stated formula (6.43). □

6.6 Generalized Hurwitz–Lerch Zeta Gamma distribution

Gamma–type distributions, associated with certain special functions of science and engineering, are studied by several researchers, such as Stacy [122]. In this section a new probability density function is introduced, which extends both the well–known Gamma distribution [106, 132] and Planck distribution [45].

Consider the r.v. X defined on a standard probability space $(\Omega, \mathfrak{F}, P)$, defined by the PDF

$$f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \dfrac{{}_2\Psi_1^*\left[\begin{matrix} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{matrix} \middle| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b)}, & \text{if } x > 0 , \\ 0, & \text{if } x \leq 0 \end{cases} \tag{6.44}$$

where a, b are scale parameters and s is shape parameter. Further $\Re(a), \Re(s) > 0$ when $|z| \leq 1$ $(z \neq 1)$ and $\Re(s) > 1$ when $z = 1$. Such distribution we call by convention *generalized Hurwitz–Lerch Zeta Gamma distribution* and write $X \sim$ HLZG. Notice that behavior of $f(x)$ near to the origin depends on s in the manner that $f(0) = 0$ for $s > 1$, and for $s = 1$ we have

$$f(0) = \frac{b \; {}_2\Psi_1^*\left[\begin{matrix} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{matrix} \middle| z \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,1,a/b)},$$

and $\lim_{x \to 0+} f(x) = \infty$ when $0 < s < 1$.

Now, we list some important special cases of the HLZG distribution.

- For $\sigma = \rho = \kappa = 1$ we obtain the following PDF discussed by Garg *et al.* [31]:

$$f_1(x) = \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \frac{{}_2F_1\left[\begin{matrix} \lambda,\mu \\ \nu \end{matrix} \middle| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}(z,s,a/b)},$$

  where $\Re(a), \Re(b), \Re(s) > 0$ and $|z| < 1$ or $|z| = 1$ with $\Re(\nu - \lambda - \mu) > 0$.

- If we set $\sigma = \rho = \kappa = 1$, $b = a$, $\lambda = 0$, then (6.44) reduces to the Gamma distribution [45, p. 32] and

- for $\sigma = \rho = \kappa = 1$, $\mu = \nu$, $\lambda = 1$ it reduces to the generalized Planck distribution defined by Nadarajah and Kotz [71], which is a generalization of the Planck distribution [45, p. 273].

6.6.1 Statistical functions for the HLZG distribution

In this section we will derive the statistical functions for the r.v. $X \sim$ HLZG distribution associated with PDF (6.44). For the moments m_r of fractional order $r \in \mathbb{R}_+$ we derive by definition

$$m_r(s) = \int_0^\infty x^r f(x)dx = \frac{(s)_r}{b^r}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+r, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)}. \tag{6.45}$$

Next we present the Laplace and the Fourier transforms of the probability density function (6.44), that is the generating function $G_Y(t)$ and the related CHF $\phi_Y(t)$:

$$\begin{aligned} G_Y(t) &= \mathsf{E}\, e^{-tY} = \frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, (a+t)/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)}, \\ \phi_Y(t) &= G_Y(-it) = \mathsf{E}\, e^{itY} = \frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, (a-it)/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)}. \end{aligned}$$

The second set of the statistical functions include the hazard function h and the mean residual life function K.

Theorem 6.14. (R. K. Saxena, T. K. Pogány, R. Saxena and D. Jankov [107]) *Let* $X \sim$ HLZG. *Then we have:*

$$\begin{aligned} h(x) &= \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)}\,\frac{{}_2\Psi_1^*\left[\begin{matrix}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{matrix}\,\Big|\, z e^{-bx}\right]}{\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a, b|x)} \\ K(x) &= \frac{b^s}{\Gamma(s)\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a, b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}}\,\frac{\Gamma(s+1, (a+bn)x)}{(a+bn)^{s+1}}\,\frac{z^n}{n!} - x\,. \end{aligned} \tag{6.46}$$

Here

$$\Gamma(p, z) = \int_z^\infty t^{p-1} e^{-t}\, dt\,, \qquad \Re(p) > 0\,,$$

stands for the upper incomplete Gamma function.

Proof. From the hazard function formula a simple calculation gives:

$$\begin{aligned} K(x) &= \frac{b^s}{\Gamma(s)\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a, b|x)} \int_x^\infty t^s e^{-at}\, {}_2\Psi_1^*\left[\begin{matrix}(\lambda,\rho),\,(\mu,\sigma)\\(\nu,\kappa)\end{matrix}\,\Big|\, z e^{-bt}\right] dt - x \\ &= \frac{b^s}{\Gamma(s)\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a, b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}}\,\frac{z^n}{n!}\int_x^\infty t^s e^{-(a+bn)t}\, dt - x\,. \end{aligned}$$

Further simplification leads to the asserted formula (6.46). □

6.7 Statistical parameter estimation in HLZB$'$ and HLZG distribution models

The statistical parameter estimation becomes one of the main tools in random model identification procedures. In studied HLZB$'$ and HLZG distributions the PDFs (6.38) and (6.44) are built by higher transcendental functions such as generalized Hurwitz–Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)$ and Fox–Wright generalized hypergeometric function ${}_2\Psi^*_1[z]$. The power series definitions of these functions does not enable the successful implementation of the popular and efficient Maximum Likelihood (ML) parameter estimation, only the numerical system solving can reach any result for HLZB$'$, while ML cannot be used for HLZG distribution case, being the extrema of the likelihood function out of the parameter space.

Therefore, we consider the Moment Method estimators, such that are weakly consistent (by the Khinchin's Law of Large Numbers [102]), also strongly consistent (by the Kolmogorov LLN [25]) and asymptotically unbiased.

6.7.1 Parameter estimation in HLZB$'$ model

Assume that the considered statistical population possesses HLZB$'$ distribution, that is the r.v. $X \sim f(x)$, (6.38) generates n independent, identically distributed replicæ $\Xi = \big(X_j\big)_{j=\overline{1,n}}$ which forms a statistical sample of the size n. We are now interested in estimating the 9-dimensional parameter

$$\theta_9 = (a, \sigma, \kappa, \rho, \lambda, \mu, \nu, z, s)$$

or some of its coordinates by means of the sample Ξ.

First we consider the PDF (6.38) for small $z \to 0$. For such values we get asymptotic

$$f(x) \sim \frac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^{\nu}}\,, \qquad x > 0,$$

which is the familiar Beta distribution of the second kind (or Beta prime) $\mathrm{B}'(\lambda,\nu)$. The moment method estimators for the remaining parameters $\lambda > 0, \nu > 2$ read:

$$\widetilde{\lambda} = \frac{\overline{X}_n\big(\overline{X^2_n} + \overline{X}_n\big)}{\overline{S}^2_n}, \quad \widetilde{\nu} = \frac{\overline{X^2_n} + \overline{X}_n}{\overline{S}^2_n}\big(\overline{X}_n + 1\big) + 1\,,$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j, \qquad \overline{S}^2_n = \frac{1}{n}\sum_{j=1}^{n}\big(X_j - \overline{X}_n\big)^2$$

expressing the sample mean and the sample variance respectively. Let us mention that for $\nu < 2$, the variance of a r.v. $X \sim \mathrm{B}'(\lambda,\nu)$ does not exists, so for these range of parameters MM is senseless.

The case of full range parameter estimation is highly complicated. The moment method estimator can be reached by virtue of the positive integer order moments formula (6.41) substituting

$$\overline{X_n^r} = \frac{1}{n}\sum_{j=1}^{n} X_j^r \mapsto m_r,$$

where $\overline{X_n^r}$ is the rth sample moment. Thus, numerical solution of the system

$$\frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r}\,\frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} = \overline{X_n^r}\,, \qquad r=\overline{1,9}$$

which results in the vectorial moment estimator $\widetilde{\theta}_9 = (\widetilde{a},\widetilde{\sigma},\widetilde{\kappa},\widetilde{\rho},\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu},\widetilde{z},\widetilde{s})$.

6.7.2 Parameter estimation in HLZG distribution

To achieve Gamma distribution's PDF from the density function (6.44) of HLZG in a way different then the second special case derived in Section 6.6, it is enough to consider the PDF (6.44) for $a = b$ and small $z \to 0$. Indeed, we have

$$\lim_{z\to 0} f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-bx}}{\Gamma(s)}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}.$$

It is well known that the moment method estimators for parameters b, s are

$$\widetilde{b} = \frac{\overline{X}_n}{\overline{S}_n^2}, \qquad \widetilde{s} = \frac{\left(\overline{X}_n\right)^2}{\overline{S}_n^2}\,.$$

The general case includes the vectorial parameter

$$\theta_{10} = (a, b, s, \lambda, \rho, \mu, \sigma, \nu, \kappa, z)\,.$$

First we show a kind of recurrence relation for the fractional order moments between distant neighbors.

Theorem 6.15. (R. K. Saxena, T. K. Pogány, R. Saxena and D. Jankov [107]) *Let $0 \le t \le r$ be nonnegative real numbers, and m_r denotes the fractional positive rth order moment of a r.v. $X \sim$ HLZG. Then it holds true*

$$m_r(s) = m_{r-t}(s+t)\cdot m_t(s)\,, \qquad s>0,\ 0\le t\le r\,. \tag{6.47}$$

Proof. It is not difficult to prove

$$\begin{aligned} m_r(s) &= \frac{(s)_r}{b^r}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+r,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b)} \\ &= \frac{\Gamma(s+r)}{b^{r-t}\,\Gamma(s+t)}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+r,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+t,a/b)}\,\frac{\Gamma(s+t)}{b^t\,\Gamma(s)}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+t,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b)}\,, \end{aligned}$$

which is equivalent to the assertion of the Theorem 6.15. □

Remark 6.16. Taking the integer order moments (6.45), that is $m_r, r \in \mathbb{N}_0$, the recurrence relation (6.47) becomes a contiguous relation for distant neighbors:

$$m_\ell(s) = m_{\ell-k}(s+k)\cdot m_k(s) = \frac{(s+k)_{\ell-k}}{b^{\ell-k}}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+\ell,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+k,a/b)}\,m_k(s) \tag{6.48}$$

for all $0 \le k \le \ell$, $k,\ell \in \mathbb{N}_0$. ∎

Choosing a system of 10 suitable different equations like (6.47) in which m_r is substituted with $\overline{X^r_n} \mapsto m_r$, we get

$$\frac{(s+t)_{r-t}}{b^{r-t}}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+r,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+t,a/b)} = \frac{\overline{X^r_n}}{\overline{X^t_n}}\,. \tag{6.49}$$

However, the at least complicated case of (6.47) occurs at the contiguous (6.48) with $k=0, \ell=\overline{1,10}$, that is, by virtue of (6.49) we deduce the system in unknown θ_{10}:

$$(s)_\ell\,\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+\ell,a/b) = b^\ell\,\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b)\,\overline{X^\ell_n}\,, \qquad \ell=\overline{1,10}\,. \tag{6.50}$$

The numerical solution of system (6.50) with respect to unknown parameter vector θ_{10} we call moment method estimator $\widetilde{\theta}_{10}$.

Remark 6.17. From (6.47) we can easily get the following recursive relation

$$m_r(s) = m_{r-1}(s+1)\cdot m_1(s) = m_{r-2}(s+2)\cdot m_1(s+1)\cdot m_1(s) = \cdots = m_{\{r\}}(s+[r])\cdot\prod_{j=0}^{[r]} m_1(s+j)\,.$$

From the previous relation, for $r \in \mathbb{N}_0$, it obviously holds

$$m_r(s) = \prod_{j=0}^{r} m_1(s+j)\,. \tag{6.51}$$

Now, from (6.45) and (6.51), we can derive a new formula for the r–th moment of HLZG distribution:

$$m_r(s) = \prod_{j=0}^{r}\frac{s+j}{b}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s+j+1,a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b)}\,, \qquad r\in\mathbb{N}_0\,.$$

∎

Conclusion

In this paper, the main theme of our research were integral representations of functional series of hypergeometric and Bessel types, and the two-sided inequalities of the mentioned hypergeometric functions.

The research consists of several separate parts, which are connected into a harmonious whole, by the obtained results.

Results, which are derived in the thesis, give a large contribution to the theory of special functions, which has been developed since the 18th century.
We give special attention to the Bessel functions of the first kind. Namely, It is well known that there are three types of functional series with members containing Bessel functions of the first kind. Those are Neumann, Schlömilch and Kapteyn series.
In the thesis, there are results which contain new integral representations of the above mentioned series and also of the modified Neumann series of the first and second kind, which we define too.
Contribution of the thesis in the field of functional series of Bessel types is reflected also in the determination of coefficients of Neumann series of Bessel functions of the first kind, which had been the open problem since 2009, posed by Pogány [92].

To the best of our knowledge, until now there are not known inequalities for Hurwitz-Lerch Zeta function. So, in the important contributions of the thesis, we can also state the two-sided inequalities for the extended general Hurwitz-Lerch Zeta function and the extended Hurwitz-Lerch Zeta function, derived from their integral representations.

In addition to the above results, new ideas and results appear and they provide a space for further development of the theory of functional series.

The next step would be to unify the theory of summing of functional series, whose members can be expressed by the nonhomogeneous differential equations, so that the nonhomogeneous part becomes associated with the initial functional series.
If we have integral representation of the initial series, we could derive new results at the similar way as we did it e.g. in the Section 3.2.1, which contains some results connected with the Neumann series, and where we get the function $\mathfrak{P}_\nu(x)$, which defines the so–called Neumann series of Bessel functions associated to the Neumann series $\mathfrak{N}_\nu(z)$.

Zaključak

U ovom radu, glavna tema našeg istraživanja bile su integralne reprezentacije funkcionalnih redova hipergeometrijskog i Besselovog tipa, te dvostrane nejednakosti pomenutih hipergeometrijskih funkcija.

Istraživanje se sastoji od nekoliko zasebnih dijelova, koje dobiveni rezultati povezuju u skladnu cjelinu.

Rezultati navedeni u tezi, u velikoj mjeri pridonose razvoju teorije specijalnih funkcija, čije početke nalazimo još u 18. stoljeću.
Posebnu pažnju pridajemo Besselovim funkcijama prve vrste. Naime, poznato je da postoje tri vrste funkcionalnih redova čiji članovi sadrže Besselove funkcije prve vrste. To su Neumannovi, Schlömilchovi, te Kapteynovi redovi.
U tezi navodimo rezultate koji sadrže nove integralne reprezentacije pomenutih redova, kao i modificiranih Neumannovih redova prve i druge vrste, koje također definiramo.
Doprinos teze u području funkcionalnih redova Besselovog tipa ogleda se i u određivanju koeficijenata Neumannovog reda Besselovih funkcija prve vrste, što je bio otvoren problem iz 2009. godine, postavljen od strane Poganja [92].

Napominjemo da do sada nisu bile poznate nejednakosti za Hurwitz–Lerch Zeta fukciju. Dakle, u važnije doprinose teze možemo uvrstiti i dvostrane nejednakosti poopćene Hurwitz–Lerch Zeta funkcije, te proširene Hurwitz–Lerch Zeta funkcije, koje dobivamo korištenjem njihovih integralnih reprezentacija.

Pored navedenih rezultata javljaju se nove ideje i neriješeni problemi koji daju prostora za daljnji razvoj problematike funkcionalnih redova.

Sljedeći korak u istraživanju može biti nalaženje generalne metode sumiranja funkcionalnih redova čiji članovi zadovoljavaju nehomogenu diferencijalnu jednadžbu, tako da nehomogeni dio postane pridružen polaznom funkcionalnom redu.
Štoviše, ako znamo integralnu reprezentaciju polaznog funkcionalnog reda, tada možemo izvesti nove rezultate na sličan način kao što smo to napravili npr. u poglavlju 3.2.1, koje sadrži rezultate vezane uz Neumannove redove, te je dobivena i funkcija $\mathfrak{P}_\nu(x)$, što je Neumannov red Besselovih funkcija povezan s $\mathfrak{N}_\nu(z)$.

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Curriculum Vitae

I was born on 5 January 1985, in Vukovar. Elementary school I attended in Borovo and secondary school in Vukovar. In 2003 I enroled the graduate study in Mathematics and Computer Science at the Department of Mathematics, University of Osijek, where I graduated in September 2007. Work on my diploma thesis "Method of moments" was supervised by M. Benšić.

From 1 November 2007, I am employed at Department of Mathematics, Josip Juraj Strossmayer University of Osijek, as a teaching assistant. During these four years I participated in teaching several undergraduate and graduate courses.

In 2007 I also enroled the Postgraduate study in Mathematics at the Department of Mathematics, University of Zagreb. During my postdoctoral studies I started the collaboration with Tibor Pogány from Faculty of Maritime Studies, University of Rijeka. This collaboration resulted in the following papers:

- Á. Baricz, D. Jankov, T.K. Pogány, Integral representations for Neumann-type series of Bessel functions I_ν, Y_ν and K_ν, *Proc. Amer. Math. Soc.* (2011) (to appear).
- Á. Baricz, D. Jankov, T. K. Pogány, Integral representation of first kind Kapteyn series, *J. Math. Phys.* **52(4)** (2011), Art. 043518, pp. 7.
- Á. Baricz, D. Jankov, T. K. Pogány, On Neumann series of Bessel functions, *Integral Transforms Spec. Funct.* (2011) (to appear).
- D. Jankov, T. K. Pogány, Integral representation of Schlömilch series (submitted).
- D. Jankov, T. K. Pogány, R. K. Saxena, An extended general Hurwitz–Lerch Zeta function as a Mathieu (a, λ) – series, *Appl. Math. Lett.* **24(8)** (2011), 1473–1476.
- D. Jankov, T. K. Pogány, E. Süli, On the coefficients of Neumann series of Bessel functions, *J. Math. Anal. Appl.* **380(2)** (2011), 628–631.
- R. K. Saxena, T. K. Pogány, R. Saxena, D. Jankov, On generalized Hurwitz-Lerch Zeta distributions occuring in statistical inference, *Acta Univ. Sapientiae Math.* **3(1)** (2011), 43–59.
- H. M. Srivastava, D. Jankov, T. K. Pogány, R. K. Saxena, Two-sided inequalities for the extended Hurwitz–Lerch Zeta function, *Comput. Math. Appl.* **62(1)** (2011), 516–522.

Besides these papers, for publication are also accepted

- D. Jankov, S. Sušić, *Geometric median in the plane*, Elem. Math. (2010) (to appear).
- D. Jankov, *Egipatski razlomci*, Osječki matematički list (2011) (to appear).

During my postgraduate education I gave presentations at the following conferences:

- D. Jankov, M. Benšić, *Parameter estimation for a three–parameter Weibull distribution –a comparative study*, 12th International Conference on Operational Research, Pula, 2008.
- D. Jankov, R. K. Saxena, T. K. Pogány, *On generalized Hurwitz–Lerch Zeta distributions occuring in statistical inference*, Second conference of the Central European Network, Zürich, 2011.

I also gave a lecture on the following seminars:

- Seminar on Optimization and Applications, Department of Mathematics, University of Osijek, Osijek.
- Numerical Mathematics and Scientific Computing Seminar, Department of Mathematics, University of Zagreb, Zagreb.
- Seminar on Inequalities and Applications, Faculty of Electrical Engineering and Computing, Zagreb.
- Seminar on Topology, Department of Mathematics, University of Zagreb, Zagreb.

Životopis

Rođena sam 05.01.1985. u Vukovaru. Osnovnu školu sam pohađala u Borovu, a srednju školu u Vukovaru. 2003. godine upisala sam se na Odjel za matematiku, sveušilišta Josipa Jurja Strossmayera u Osijeku, smjer matematika–informatika. Diplomirala sam u rujnu, 2007. godine, s temom diplomskog rada "Metoda momenata", pod mentorstvom M. Benšić.

Od 1. studenog, 2007., zaposlena sam na Odjelu za matematiku, Sveučilišta J. J. Strossmayera u Osijeku, kao asistent. Tijekom te četiri godine držala sam vježbe iz više dodiplomskih i diplomskih predmeta.

2007. sam također upisala i Poslijediplomski studij matematike, na Matematičkom odsjeku, Sveučilišta u Zagrebu. Tijekom poslijediplomskog studija započela sam suradnju s Tiborom Poganjem, s Pomorskog fakulteta, Sveučilišta u Rijeci. Ova je suradnja rezultirala sljedećim člancima:

- Á. Baricz, D. Jankov, T.K. Pogány, Integral representations for Neumann-type series of Bessel functions I_ν, Y_ν and K_ν, *Proc. Amer. Math. Soc.* (2011) (to appear).
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- H. M. Srivastava, D. Jankov, T. K. Pogány, R. K. Saxena, Two-sided inequalities for the extended Hurwitz–Lerch Zeta function, *Comput. Math. Appl.* **62(1)** (2011), 516–522.

Pored navedenih članaka, za objavljivanje su prihvaćeni i sljedeći:

- D. Jankov, S. Sušić, *Geometric median in the plane*, Elem. Math. (2010) (to appear).
- D. Jankov, *Egipatski razlomci*, Osječki matematički list (2011) (to appear).

Tijekom posljediplomskog studija, izlagala sam na dvije konferencije:

- D. Jankov, M. Benšić, *Parameter estimation for a three–parameter Weibull distribution –a comparative study*, 12th International Conference on Operational Research, Pula, 2008.

- D. Jankov, R. K. Saxena, T. K. Pogány, *On generalized Hurwitz–Lerch Zeta distributions occuring in statistical inference*, Second conference of the Central European Network, Zürich, 2011.

te sam također održala predavanja na sljedećim seminarima:

- Seminar za optimizaciju i primjene, Odjel za matematiku, Sveučilište u Osijeku, Osijek.
- Seminar za numeričku matematiku i računarstvo, Matematički odsjek, Sveučilište u Zagrebu, Zagreb.
- Seminar za nejednakosti i primjene, Fakultet elektrotehnike i računarstva, Zagreb.
- Seminar za topologiju, Matematički odsjek, Sveučilište u Zagrebu, Zagreb.

Summary

This thesis presents some new results on integral expressions for series of functions of hypergeometric and Bessel types. Also there are derived two–sided inequalities of some hypergeometric functions, which are related with their integral representations.

In the first part of the thesis are defined some special functions, mathematical methods, and results which we use in prooving our own. Some of them are Gamma function, Gauss hypergeometric function ${}_2F_1$ and generalized hypergeometric function ${}_pF_q$. There are also Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ and the Struve function $\mathbf{H}_\nu(z)$.
Bessel differential equation is also described, and that is one of the crucial mathematical tools that we use.
Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$– and Dirichlet series are defined too, because they are useful for deriving most of integral representations. In that purpose, we also use condensed form of Euler–Maclaurin summation formula and fractional analysis, which are described in the introduction.

In the middle part of the thesis, i.e. in Chapter 3, 4 and 5 we work on integral representations of functional series with members containing Bessel functions of the first kind, which are divided into three types: Neumann series, which are discussed in Chapter 3, Kapteyn series, which are described in Chapter 4, and Schlömilch series, which are observed in Chapter 5.

In the last chapter of this thesis, we obtain a functional series of hypergeometric types. There, we also derive an integral representations of hypergeometric functions, such as extended general Hurwitz–Lerch Zeta function and extended Hurwitz–Lerch Zeta function, and also the two-sided inequalities for the mentioned functions.
At the end of this chapter, new incomplete generalized Hurwitz–Lerch Zeta functions and incomplete generalized Gamma functions are defined, and we also investigate their important properties.

Sažetak

U ovoj disertaciji dani su rezultati vezani uz predstavljanje funkcionalnih redova hipergeometrijskog i Besselovog tipa integralom. Također su izvedene i dvostrane nejednakosti pojedinih hipergeometrijskih funkcija, koje su usko vezane s integralnim reprezentacijama istih.

U prvom su dijelu rada najprije definirane specijalne funkcije, matematiki alati, te rezultati koje koristimo pri dokazivanju vlastitih. Neke od njih su Gama funkcija, Gaussova hipergeometrijska funkcija ${}_2F_1$, te njezina generalizacija ${}_pF_q$, kao i Fox–Wrightova generalizirana hipergeometrijska funkcija ${}_p\Psi_q$, te Struveova funkcija $\mathbf{H}_\nu(z)$.
Opisana je i Besselova diferencijalna jednadžba, koja nam je jedan od glavnih matematičkih alata.
Definirani su i Mathieuovi $(\boldsymbol{a}, \boldsymbol{\lambda})$–, te Dirichletovi redovi, koje koristimo prilikom izvođenja većine integralnih reprezentacija. U tu svrhu koristimo i kondenzirani oblik Euler–Maclaurinove sumacijske formule, te frakcionalnu analizu čiji opis također navodimo u uvodnom dijelu.

U središnjem dijelu rada, tj. u poglavljima 3, 4 i 5 bavimo se integralnim reprezentacijama funkcionalnih redova Besselovog tipa od kojih postoje tri tipa: Neumannovi redovi, koje promatramo u poglavlju 3, Kapteynovi redovi, koji su opisani u poglavlju 4, te na kraju Schlömilchovi redovi, čije integralne reprezentacije izvodimo u poglavlju 5.

U šestom, ujedno i posljednjem poglavlju, promatramo funkcionalne redove hipergeometrijskog tipa. Izvode se integralne reprezentacije hipergeometrijskih funkcija kao što su poopćena Hurwitz-Lerch Zeta i proširena Hurwitz-Lerch Zeta funkcija, te dvostrane nejednakosti navedenih funkcija.
Na kraju ovog poglavlja bavimo se poopćenom Hurwitz-Lerch Zeta distribucijom, te definiramo nove nepotpune, generalizirane Hurwitz–Lerch Zeta funkcije i nepotpune generalizirane Gamma funkcije, za koje također ispitujemo i osnovna svojstva.