Time-changed SIRV epidemiological model

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Abstract

The stochastic version of the SIRV (susceptible-infected-recovered-vaccinated) model in the population of non-constant size and finite period of immunity is considered. Among many parameters, the most important is the contact rate, i.e. the average number of adequate contacts of an infective person. It is expected that this parameter exhibits time-space clusters which is reflected in interchanging periods of low and steady transmission and periods of high and volatile transmission of the disease.

The stochastics in the SIRV model considered here comes from the noise represented as the sum of the conditional Brownian motion and Poisson random field, closely related to the corresponding time-changed Brownian motion and the time-changed Poisson random measure.

The existence and uniqueness of positive global solution of the stochastic SIRV process is proven by classical techniques. Furthermore, persistence and extinction of infection in population in long-run scenario are analyzed.

Deterministic SIRV model

Population is divided into **four mutually exclusive compartments**:

- **S** - susceptible individuals,
- **I** - infected individuals,
- **R** - recovered individuals,
- **V** - vaccinated individuals.

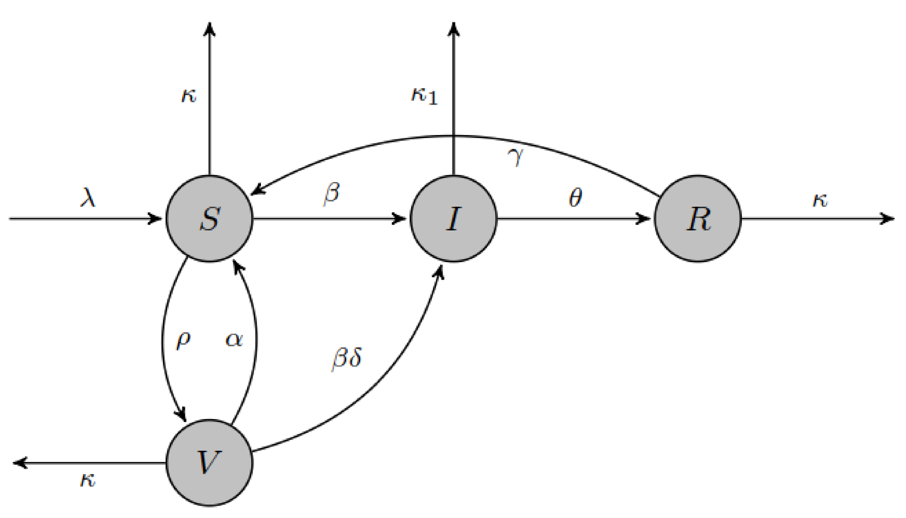


Figure 1. Scheme of the SIRV model with vaccination and temporary immunity.

- **Total population size** at time $t \geq 0$:

$$N(t) = S(t) + I(t) + R(t) + V(t) < K.$$

- $K \in \mathbb{R}_+$ - **the carrying capacity of the ecosystem.**
- Deterministic SIRV model - **system of ODEs**:

$$\begin{aligned} dS(t) &= \left(\left(\lambda - \kappa - \rho - \tfrac{\beta}{N(t)} I(t)\right) S(t) + \alpha V(t) + \gamma R(t)\right) dt \\ dI(t) &= \left(\tfrac{\beta}{N(t)} I(t)\,(S(t) + \delta V(t)) - (\kappa_1 + \theta) I(t)\right) dt \\ dR(t) &= (\theta I(t) - (\kappa + \gamma) R(t))\ dt \\ dV(t) &= \left(\rho S(t) - (\kappa + \alpha + \tfrac{\delta\beta}{N(t)} I(t)) V(t)\right) dt. \end{aligned}$$

Modeling of the contact rate β

- Modeling of the contact rate β - based on the **time-changed Lévy noise** introduced in **Di Nunno, G. & Sjursen, S. (2014)**.
- **Model for contact rate β** - driven by the random measure μ:

$$\beta dt \mapsto \beta dt + \int_{\mathbb{R}} \sigma_t(z) \mu(dt, dz).$$

- **Measure μ** - the mixture of a **conditional Brownian measure** B on $[0,T] \times \{0\}$ and a **centered doubly stochastic Poisson measure** $\widetilde{H}$ on $[0,T] \times \mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$.

Definition and properties of measure μ

- $(\Omega, \mathcal{F}, \mathbb{P})$ - a complete probability space.
- $X = [0,T] \times \mathbb{R} = ([0,T] \times \{0\}) \cup ([0,T] \times \mathbb{R}_0), \ \ T > 0.$
- $\mathcal{B}_X$ - Borel σ-algebra on X.
- $\Delta \in \mathcal{B}_X$ - a Borel subset of X.
- $\lambda := \left(\lambda^B, \lambda^H\right)$ - a two dimensional stochastic process such that each component λ^l, $l = B, H$, satisfies:
  (i) $\lambda_t^l \geq 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$,
  (ii) $\lim_{h\to 0} \mathbb{P}\left(\left|\lambda_{t+h}^l - \lambda_t^l\right| \geq \varepsilon\right) = 0$ for all $\varepsilon > 0$ and almost all $t \in [0,T]$,
  (iii) $\mathbb{E}\left[\int_0^T \lambda_t^l\, dt\right] < \infty$.
- **Random measure Λ** on X:

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t) \lambda_t^B\, dt + \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\Delta}(t,z) \nu(dz) \lambda_t^H\, dt,$$

where ν is a deterministic, σ-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.$$

- If $\Lambda^B(\Delta)$ is the restriction of Λ to $[0,T] \times \{0\}$ and $\Lambda^H(\Delta)$ the restriction of Λ to $[0,T] \times \mathbb{R}_0$, then:

$$\Lambda(\Delta) = \Lambda^B(\Delta \cap [0,T] \times \{0\}) + \Lambda^H(\Delta \cap [0,T] \times \mathbb{R}_0).$$

Definition 1 (Di Nunno & Sjursen, 2014). ***The random measure*** μ *on the Borel subsets of* X *is defined by*

$$\mu(\Delta) := B(\Delta \cap [0,T] \times \{0\}) + \widetilde{H}(\Delta \cap [0,T] \times \mathbb{R}_0), \quad \Delta \subseteq X,$$

where

- *conditionally on* Λ, B **is a Gaussian random measure**,
- *conditionally on* Λ, H **is a Poisson random measure**,
- $\widetilde{H} := H - \Lambda^H$ *is a measure given by*

$$\widetilde{H}(\Delta) := H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}_0.$$

The model for contact rate β can be written in the following form:

$$\beta dt \mapsto \beta dt + \sigma_t(0) dB_t + \int_{\mathbb{R}_0} \sigma_t(z) \widetilde{H}(dt, dz).$$

Random measures B and H are related to a specific form of **time-change** for Brownian motion and pure jump Lévy process:

$$B_t := B([0,t] \times \{0\}), \quad \Lambda_t^B := \int_0^t \lambda_s^B\, ds, \quad t \in [0,T],$$

$$\eta_t := \int_0^t \int_{\mathbb{R}_0} z \tilde{H}(ds, dz), \quad \widehat{\Lambda}_t^H := \int_0^t \lambda_s^H\, ds, \quad t \in [0,T].$$

Theorem 1 (Serfozo, 1972). *Let* $W = (W_t, t \in [0,T])$ *be a Brownian motion and* $N = (N_t, t \in [0,T])$ *be a centered pure jump Lévy process with Lévy measure* ν. *Assume that both* W *and* N *are independent of* Λ. *Conditionally on* Λ:

- B **is a Gaussian random measure** *such that* $B(\Delta_1)$ *and* $B(\Delta_2)$ *are independent whenever* $\Delta_1 \cap \Delta_2 = \emptyset$, **if and only if for any** $t \geq 0$

$$B_t \overset{d}{=} W_{\Lambda_t^B},$$

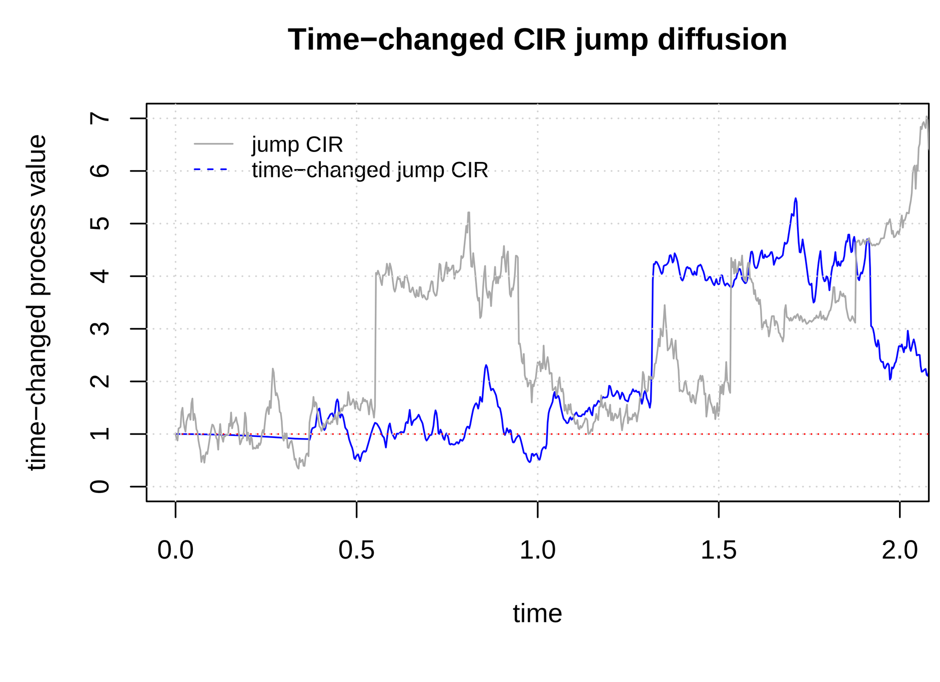
- η **is a Poisson random measure**, *independent on* B *and such that* $H(\Delta_1)$ *and* $H(\Delta_2)$ *are independent whenever* $\Delta_1 \cap \Delta_2 = \emptyset$, **if and only if for any** $t \geq 0$

$$\eta_t \overset{d}{=} N_{\hat{\Lambda}_t^H}.$$

Example model for contact rate β
Time-changed CIR jump diffusion

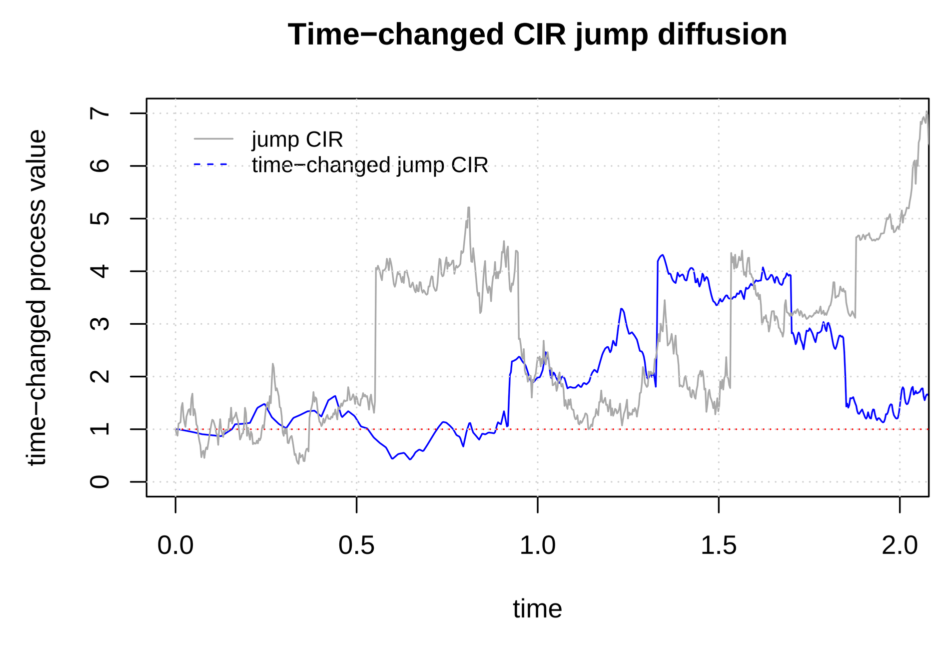
Example 1.

- $\lambda_t^B = \lambda_t^H = at + \sum_{k=0}^{N_t} X_k$ - **CPP with drift** for $a = 0.05$, $X_t \sim \mathcal{U}(-1, 0.6)$ and $(N_t,\ t \geq 0)$ a PP with intensity $\lambda = 2$.



Example 2.

- λ_t^B - **IG(α, δ) subordinator** for $\alpha = 1$, $\delta = 5$
- $d\lambda_t^H = -\theta(\lambda_t^H - \mu)dt + \sigma dB_t$ - for $\theta = 5$, $\mu = 0$, $\sigma = 3$ **mean-reverting Ornstein-Uhlenbeck process**.



SIRV model driven by random measure μ

$$\begin{aligned} dS(t) &= \left((\lambda - \rho - \kappa) S(t) - \frac{\beta}{N(t)} S(t) I(t) + \alpha V(t) + \gamma R(t)\right) dt \\ &\quad - \int_{\mathbb{R}} \sigma_t(z) \frac{S(t)}{N(t)} I(t)\, \mu(dt, dz) \\ dI(t) &= \left(\frac{\beta}{N(t)} (S(t) + \delta V(t)) - (\kappa_1 + \theta)\right) I(t)\, dt \\ &\quad + \int_{\mathbb{R}} \sigma_t(z)\, (S(t) + \delta V(t))\, \frac{I(t)}{N(t)}\, \mu(dt, dz) \\ dR(t) &= (\theta I(t) - (\kappa + \gamma) R(t))\ dt \\ dV(t) &= \left(\rho S(t) - (\kappa + \alpha) V(t) - \delta \frac{\beta}{N(t)} V(t) I(t)\right) dt \\ &\quad - \int_{\mathbb{R}} \sigma_t(z) \delta \frac{V(t)}{N(t)} I(t)\, \mu(dt, dz) \end{aligned}$$

Existence of unique positive global solution

Theorem 2. *For any initial value* $(S(0), I(0), R(0), V(0)) \in \langle 0, K \rangle^4$ *there exist a* **unique positive global solution**

$$((S(t), I(t), R(t), V(t)), t \geq 0)$$

of the SIRV SDE system that $\mathbb{P}$**-a.s. remains in** $\langle 0, K \rangle^4$.

Extinction theorem

Theorem 3. *If*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \frac{ds}{(\lambda_s^B \sigma_s(0))^2} < \frac{2(\kappa_1 + \theta)}{K^2}, \ \ \mathbb{P} - a.s.,$$

$$\lim_{t\to\infty} \int_0^t \frac{1}{(1+s)^2} \int_{\mathbb{R}} \left(\lambda_s^B 1_{\{0\}}(z) + \lambda_s^H 1_{\mathbb{R}_0}(z)\right) \nu(dz) ds < \infty,$$

then for any initial value $(S(0), I(0), R(0), V(0)) \in \langle 0, K \rangle^4$

$$\mathbf{I(t) \to 0} \quad \mathbb{P} - a.s.\ as\ \mathbf{t} \to \infty,$$
$$\mathbf{R(t) \to 0} \quad \mathbb{P} - a.s.\ as\ \mathbf{t} \to \infty,$$

while

$$\limsup_{\mathbf{t}\to\infty} (\mathbf{S(t) + V(t)}) = \mathbf{K} \quad \mathbb{P} - \mathbf{a.s.}$$

Persistence theorem

The system is said to be persistent in the mean if

$$\liminf_{t\to\infty} [I(t)] := \liminf_{t\to\infty} \frac{1}{t} \int_0^t I(s) ds > 0, \quad \mathbb{P} - a.s.$$

Theorem 4. *For any initial value* $(S(0), I(0), R(0), V(0)) \in \langle 0, K \rangle^4$

$$\liminf_{\mathbf{t}\to\infty} [\mathbf{I(t)}] > \mathbf{0} \quad \mathbb{P} - a.s.$$

if the following conditions are satisfied:

- $\lambda > \rho$,
- *there exists a positive constant* $\widetilde{\beta}$ *such that* $\liminf_{t\to\infty} \frac{\beta}{N(t)} \geq \widetilde{\beta}$
- $\limsup_{t\to\infty} \frac{1}{t} \int_0^t \sigma_s^2(0) (\lambda_s^B)^2 ds \leq \frac{2\widetilde{\beta}(\lambda + \rho(\delta - 1))\underline{S} + \alpha(1-\delta)\underline{V})}{C\kappa(\delta+1)^2}$,
  where $\underline{V}$ *is such that* $V(t) \geq \underline{V}$ *for all* $t \geq 0$ *and* $C := \kappa_1 + \theta - \underline{V}\widetilde{\beta}\delta(1-\delta) - \frac{\theta\gamma}{\kappa+\gamma} > 0$,
- $\lim_{t\to\infty} \int_0^t \frac{1}{(1+s)^2} \int_{\mathbb{R}} \left(\lambda_s^B \mathbf{1}_{\{0\}}(z) + \lambda_s^H \mathbf{1}_{\mathbb{R}_0}(z)\right) \nu(dz) ds < \infty.$

Forthcoming research

- Simulation of the stochastic SIRV system in extinction and persistence scenarios, regarding different time-changed models for contact rate β driven by random measure μ.
- Recovering the time-dependent transmission rate from infection data via solution of an inverse problem.

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