Bi-Log-Concave Distribution Functions and Confidence Bands

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Abstract

In nonparametric statistics one is often interested in estimators or confidence regions for curves such as densities or regression functions. Estimation of such curves is typically an ill-posed problem and requires additional assumptions. An interesting alternative to smoothness assumptions are qualitative constraints, e.g. monotonicity, concavity or log-concavity. Estimation of a distribution function F based on independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ with c.d.f. F is less difficult. But non-trivial confidence regions for certain functionals of F such as the mean do not exist without substantial additional constraints (Bahadur and Savage, 1956).

In density estimation, a particular constraint which attracted considerable attention recently is log-concavity. That means, we estimate a probability density f on $\mathbb{R}^d$ under the constraint that $\log f : \mathbb{R}^d \to [-\infty, \infty)$ is a concave function. While many papers are focussing on point estimation, Schuhmacher et al. (2011) show that combining the log-concavity constraint and a standard Kolmogorov-Smirnov confidence region yields an interesting nonparametric confidence region, although its explicit computation is far from obvious. In the present work we introduce a new and weaker constraint on distribution functions:

A distribution function F on the real line is called *bi-log-concave* if both $\log F$ and $\log(1 - F)$ are concave functions (with values in $[-\infty, 0]$).

This new shape constraint is rather natural in many situations. For instance, any c.d.f. F with log-concave density $f = F'$ is bi-log-concave, according to Bagnoli and Bergstrom (2005). But bi-log-concavity of F alone is a much weaker constraint: F may have a density with an arbitrarily large number of modes. Various characterizations of bi-log-concavity are provided. It is shown that combining any nonparametric confidence band for F with the new shape-constraint leads to substantial improvements and implies non-trivial confidence bounds for arbitrary moments and the moment generating function of F.

Keywords: Shape constraints, log-concavity, confidence set, Empirical distribution, Kolmogorov-Smirnov.

AMS subject classifications: 62G05, 62G15, 62G20, 62G30

Acknowledgements: This is joint work with Lutz Dümbgen (Bern) and Ralf Wilke (Nottingham).

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